Model theory of Differential Fields

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Fall 2025.

Contents

Chapter 0

Quick summary of basic model theory and Notations

Definition. A language $\mathcal{L}$ is a vocabulary including function symbols, constants, predicate symbols, and equality "=". (For now 1-sorted).

$\mathcal{L}$-structure $\mathcal{M}$ includes actual interpretations of symbols, and "=" is interpreted as equality. We identify notationally M and its universe.

$\varphi(\bar{x})$ is $\mathcal{L}$-formula with free variables included in $\bar{x}$. We say $M \models \varphi(\bar{a})$, if $\varphi(\bar{x})$ true in M, when $\bar{x}$ interpreted as $\bar{a}$, where $\bar{a} = (a_1, \ldots, a_n) \in M^n$.

given $\varphi(\bar{x}, \bar{y})$, $\bar{b} \in M$, $\{\bar{a} \in M^n : M \models \varphi(\bar{a}, \bar{b})\}$ is called a definable set, defined over (parameters) $\bar{b}$.

Remark. Look at $\varphi(\bar{x}, \bar{y})$, M is $\mathcal{L}$-structure. What can we see from here?

1. bipartite graph. vertex sets are $M^n \ni \bar{x}$ and $M^m \ni \bar{y}$. $E(\bar{a}, \bar{b})$ if $M \models \varphi(\bar{a}, \bar{b})$.

2. Definable family of definable sets:

$$\{\varphi(\bar{x}, \bar{b})(M) : \bar{b} \in M^n\}$$

   there is connection with geometry: think about moduli space in algebraic geometry. Think: can you view it as a definable set?

3. a suitable (definable) family of continuous functions. Why continuous? connection with functional analysis. We will see later.

0.1 Formalism for definability with pararmeters

Definition. Let $B \subseteq M$. Add constant c_b for all elements of B. Denote the language as $\mathcal{L}_B$. Expand M to $\mathcal{L}_B$-structures M_B tautologically. $X \subseteq M^n$ definable over B iff definable in $\mathcal{L}_B$

with no parameter.

With the notation, type $p_M(\bar{a}/B)$ = collection of $\mathcal{L}_B$-formula true of $\bar{a}$ in M.

0.2 Consistency and Compactness

In model theory there is absolute no need of proof systems.

Definition. Σ is a set of $\mathcal{L}$-sentences. Σ is consistent means Σ has a model, that is, $\exists M$, s.t. $M \models \Sigma$ i.e. $M \models \sigma$, $\forall \sigma \in \Sigma$.

Theorem (Compactness theorem). Σ *consistent iff finitely consistent: every fintie* $\Sigma' \subseteq \Sigma$ *is consistent, i.e.* $\forall \Sigma' \subseteq \Sigma$ *finite,* $\exists M \models \bigwedge \Sigma'$.

0.3 Elementary extensions and substructures

Definition. $M \prec N$ means for every L_M-sentence, $M \models \sigma$ iff $N \models \sigma$. equivalently for all $\varphi(\bar{x})$ L-formula, $\bar{a} \in M^n$, $M \models \varphi(\bar{a})$ iff $N \models \varphi(\bar{a})$.

M is L-structure. $B \subseteq M$. $\Sigma(\bar{x})$ collection of formulas over B. Σ is consistent with M, if $\exists N \succ M$ realizing Σ, i.e. $\exists \bar{a} \in N$, s.t. $N \models \varphi(\bar{b})$ for all $\varphi \in \Sigma$.

Remark. By compactness, $\Sigma(\bar{x})$ is consistent with M iff $\Sigma(\bar{x})$ is finitely satisfiable in M. i.e. for all finite $\Sigma' \subseteq \Sigma$, $M \models \exists \bar{x}(\bigwedge \Sigma'(\bar{x}))$.

0.4 Saturation

Definition. M is κ-saturated, κ infinite cardinal, means whenever $B \subseteq M$, $|B| < \kappa$ and $\Sigma(x)$ are B-consistent with M, then realized in M. ($\exists \bar{a} \in M$, $M \models \varphi(\bar{a})$, $\forall \varphi \in \Sigma$)

Example. $(\mathbb{C}, +, \times, 0, 1)$ is $2^{\aleph_0}$-saturated (and of cardinality $2^{\aleph_0}$). But $(\mathbb{R}, +, \times, -, 0, 1)$ is not $2^{\aleph_0}$-saturated. $\Sigma(x) = \{x > 0, x > 1, \dots\}$ consistent but not realized. ($\leq$ is definable by $x \leq y$ iff $\exists z(y - x = z^2)$).

Theorem. *Let* M *be a finite* L*-structure. Then* M *is* κ*-saturated, for all* κ.

Remark. Why is saturation important? Fact: if M, N, $M \equiv N$ both κ-satuated of cardinality κ, then $M \cong N$.

Given M infinite, κ a regular cardinal, $\kappa > |M|$, under GCH there is $N \succ M$ which is κ-saturated of cardinality κ.

In general with no set-theoretic assumptions, we can prove, $\forall$ infinite M and $\kappa \geq |M|$, there is $N \succ M$ which is κ-saturated and strongly κ-homogeneous: any partial elementary map f between $A, B \subseteq M$ of cardinality $< \kappa$ extends to an automorphism of M.

Fact: M κ-saturated of cardinality $\kappa \Rightarrow$ strongly κ-homogeneous.

In [5], it's explained why it's justified to assume existence of saturated model. (M saturated means κ saturated of cardinality κ).

Saturated model is a canonical model. There are also other canonical models, e.g. prime models.

0.5 n-types

Definition. Let $\mathcal{M}$ be a $\mathcal{L}$-structure. $B \subseteq M$. A *complete n-type* over B in the sense of M is a maximal collection of $\mathcal{L}_B$-formulas consistent with M. Equivalently, something of form $\text{tp}_N(\bar{a}/B)$, for some $N \succ M$, $\bar{a} \in N^n$.

Collection of such type is denoted as $S_n(B, M)$, or $S_n(B)$ if M is understood.

0.6 Theories

Definition. An $\mathcal{L}$-theory is a consistent collection of $\mathcal{L}$-sentences. (Often assumed closed under logical consequences: σ is a logical consequence of Σ, written as $\Sigma \models \sigma$, if every model of Σ is a model of σ.)

Σ is complete if for any σ, $\sigma \in \Sigma$ or $\neg\sigma \in \Sigma$.

Remark. Model theory is the study of (complete) first order theories T, and their invariants.

What do we mean "and their invariants"? From this rise two things: first-order theory, up to what? and what are some interesting invariants.

One notion, is "bi-interpretability". Invariants should depend on the bi-interpretability type.

Here are a few examples of interesting invariants:

Example. Take $\text{Mod}(T)$ = category of models of T. morphisms are elementary embeddings: $f : M \prec N$. Then there are interesting questions, e.g. can you recover T (up to bi-interpretability) from $\text{Mod}(T)$.

e.g. Shelah's work of classification is about $\text{Mod}(T)$.

Example. $\text{Def}(T)$. It's the category of definable sets. Objects are $\mathcal{L}$-formulas up to equivalence mod T, i.e. $\varphi(\bar{x}) \sim_T \psi(\bar{x})$ if $T \models (\forall\bar{x})(\varphi(\bar{x}) \leftrightarrow \psi(\bar{x}))$. Morphisms from to $[\varphi(\bar{x})]$ to $[\chi(\bar{y})]$ is given by formula $[\psi(\bar{x}, \bar{y})]$ of $\mathcal{L}$, s.t. $T \models \forall\bar{x}(\exists y\psi(\bar{x}, \bar{y}) \leftrightarrow \varphi(\bar{x})$ and $T \models \forall\bar{x}\forall\bar{y}(\psi(\bar{x}, \bar{y}) \Rightarrow \chi(\bar{y}))$, and $T \models \forall\bar{x}\exists^{=1}\bar{y}(\psi(\bar{x}, \bar{y}))$. i.e. definable functions.

Example. Various Galois groups $\text{Gal}_?(T)$.

Example. Whether or not infinite groups/fields are definable in models of T.

Example. Whether "combinatorial" structures can/cannot be definalbe in models of T.

e.g. instability is witnessed by a model $M \models T$ and bipartite definable graph (V, W, E) such that there exists $v_1, v_2, v_3, \dots \in V$, $w_1, w_2, \dots \in W$ s.t. $\forall i, j$, $v_i E w_j$ iff $i \leq j$.

e.g. $\exists$ formula $\varphi(\bar{x}, \bar{y})$ of L and model M of T and $(\bar{a}_1, \bar{a}_2, \bar{a}_3, \dots)$ in M and $\{\bar{b}_X \in M : X \subseteq \omega\}$ s.t. $M \models \varphi(a_i, b_X)$ iff $i \in X$. This we say φ has IP.

Shelah discovered that on both sides of dividing lines like these, you could prove theorems.

Remark. There are also properties of sorts in a given theory T.

E.g. M, M^2, M^3 are sorts (of singletons, pairs, triples, etc.). Strongly minimality of a sort S means S is infinite and any definable (with parameters) subset of S is finite or cofinite. It's a property of sorts, not theories.

Take a theory T and a model M, we can also look at the collection of pairs M^2 (relations are those definable without parameters in original structure) and its theory. It turns out They are bi-interpretable.

So bear in mind difference of properties of theories and sorts.

From now on let T be a fixed complete theory in $\mathcal{L}$. It's convenient to pick a big cardinal $\bar{\kappa}$ and fix some $\bar{\kappa}$-saturated, strongly $\bar{\kappa}$-homogeneous $\overline{M} \models T$. Then every model of T of cardinality $\leq \bar{\kappa}$ is an elementary substructure of $\overline{M}$ up to $\cong$. So to study T (and its invariants) we can work in $\overline{M}$.

If e.g. $\varphi(\bar{x})$ is $\mathcal{L}$-formula, $\bar{a} \in (\overline{M})^n$, write $\models \varphi(\bar{a})$ for $\overline{M} \models \varphi(\bar{a})$ ($\Leftrightarrow$ $M \models \varphi(\bar{a})$ whenever $M \prec \overline{M}$ and $\bar{a} \in M^n$).

Remark. Next time: indiscernibles, definability of sets, type spaces, examples, imaginaries and T^{eq}, definable choice and Skolem functions

Remark. Consider $(\mathbb{C}, +, \times)$ and $(\mathrm{GL}_2(\mathbb{C}), \cdot)$. They are mutually interpretable but not bi-interpretable.

Chapter 1

We start numbering

1.1 Indiscernibles

$\overline{M}$ is big model of T.

Definition 1.1. $(\bar{a}_i : i < \omega)$ a sequence of tuples of same length is *indiscernible* over small set $A \subseteq \overline{M}$, if for all $i_1 < i_2 < \cdots < i_n$, $j_1 < j_2 < \cdots < j_n$ in ω, $\text{tp}(\bar{a}_{i_1}, \ldots, \bar{a}_{i_n}/A) = \text{tp}(\bar{a}_{j_1}, \ldots, \bar{a}_{j_n}/A)$.

Lemma 1.2. *There exists such indiscernible sequences (over any A).*

Proof. Either using Ramsey or finitely satisfiable types/coheirs.

Fix small model M. ($M \prec \overline{M}$). Take $p(\bar{x}) \in S_1(M)$ nonalgebraic (not realized in M). Let $M' \succ M$ be an $|M|^+$-saturated model. Let by compactness $p'(x) \in S_1(M')$ s.t. $p(x) \subseteq p'(x)$ and $p'(x)$ is finitely satisfiable in M, i.e. each $\varphi(x)$ over M' in p', $\exists a \in M$, s.t. $\models \varphi(a)$. Using the language of French people $p'(x)$ is a coheir of p.

Consider $\Sigma(x) = \{\varphi(x) \in L_{M'}, \varphi$ not realized by some elements of $M\}$. Let $\Phi(x) = p(x) \cup \{\neg\varphi(x) : \varphi \in \Sigma\}$. It suffices to prove that $\Phi(x)$ consistent (with M'). This is because given $\psi(x) \in p$ and $\varphi_1, \ldots, \varphi_n \in \Sigma$, let $\models \psi(a)$, $a \in M$. Then $\models \neg\varphi_1(a) \wedge \ldots \wedge \neg\varphi_n(a)$. So "$\psi(x) \cup \{\neg\varphi_1(x), \ldots, \neg\varphi_n(x)\}$ is consistent. Apply "compactness", then $\Phi(x)$ is a type over M'.

So extend it to $p'(x) \in S_1(M)$ (by Zorn's lemma). (p' is special case of an "M-invariant" type). By $|M|^+$-saturation of M', let a_1 realize $p(x)$ in M'. Let a_2 realize $p' \restriction (M, a_1)$ in M'. Continue to ω we get $(a_1, a_2, \ldots)$ in M' (called Morley sequence in p' over M. An inductive argument shows $(a_1, a_2, \ldots)$ is indiscernible over M. □

Remark 1.3.

1. indiscernibility makes the same for any $(\bar{a}_i : i \in I)$ where I is an infinite totally ordered set.
2. Given a sequence $(\bar{a}_i : i \in I)$ indiscernible over A, the EM-type of the sequence over A is $\{\text{tp}(\bar{a}_{i_1}, \ldots, \bar{a}_{i_n}/A) : n < \omega,\ i_1, < \cdots < i_n \in I\}$. Then T.F.A.E.

(a) $\exists$ an infinite indiscernible over A sequence $(\bar{a}_i : i \in I)$ with given EM-type $\mathbb{P}$ (for some I).

(b) given any infinite totally ordered set I, $\exists$ indiscernible sequence over A induced by I with EM-type $\mathbb{P}$.

Proof. Compactness + saturation. □

1.2 Definability and automorphisms

Fact 1.4. Let $X \subseteq \overline{M}^n$ be definable (over some set of parameters). Let $A \subseteq M$ (small). Then

1. X is definable over A if X is fixed setwise under $\mathrm{Aut}(\overline{M}/A)$.

2. $\exists A$-definable fintie equivalence relation over $\overline{M}^n$ (i.e. finitely many classes) s.t. X is a union of some E-classes, iff X has fintiely any images under $\mathrm{Aut}(\overline{M}/A)$ iff X has $< \overline{\kappa}$ mnay images under $\mathrm{Aut}(\overline{M}/A)$. (We say X is definable almost over A).

Proof. Compactness + strongly homogeneity. □

Recall $\bar{a} \in \mathrm{acl}(A)$ if there is L_A-formula $\varphi(\bar{x})$ s.t. $\models \varphi(\bar{a})$. and $\models \exists^{=k}\bar{x}\varphi(\bar{x})$ for some $1 \leq k < \omega$. $\bar{a} \in \mathrm{dcl}(A)$ if the same as above with $k = 1$.

So special case of fact 1.4:

Fact 1.5. 1. $\bar{a} \in \mathrm{dcl}(A)$ if $\bar{a}$ is fixed under $\mathrm{Aut}(\overline{M}/A)$.

2. $\bar{a} \in \mathrm{acl}(A)$ iff $\bar{a}$ has fintiely any images under $\mathrm{Aut}(\overline{M}/A)$ iff $\bar{a}$ has $< \overline{\kappa}$ mnay images under $\mathrm{Aut}(\overline{M}/A)$.

Proof. Compactness + strongly homogeneity. □

1.3 type spaces

$B \subseteq \overline{M}$ small. We define $S_n(B, \overline{M}) = \{\mathrm{tp}_{\overline{M}}(\bar{a}/B) : \bar{a} \in \overline{M}^n)\}$. sometimes written as $S_n(B)$. When $B = \varnothing$ it's $S_n(T)$.

Definition/Fact 1.6. 1. Define $X \subseteq S_n(B)$ to be basic basic open if $\exists\varphi(\bar{x}) \in L_B$ s.t. $X = \{p(\bar{x}) \in S_n(B) : \varphi(x) \in p(\bar{x})\} = [\varphi]$. This defines a topology on $S_n(B)$, making it a compact totally disconnected space. i.e. a compact Hausdorff space with a basis of clopens. (aka a profinite space, inverse limit of finite topological spaces).

2. Using comapctness $X \subseteq S_n(B)$, clopen iff $X = [\varphi]$ for some $\varphi(\bar{x}) \in L_B$ formula.

Proof. excercise. ($\varphi(\bar{x}), \psi(\bar{x})$ over B are equiv. in $\overline{M}$ iff $[\varphi] = [\psi]$. So formulas over B, $\varphi(\bar{x})$ up to equiv. in $\overline{M}$ = clopens of $S_n(B)$.) $\square$

Corollary 1.7. *Formulas $\varphi(\bar{x})$ over B upto equivalence in $\overline{M}$ are in 1-1 correspondence with continuous functions $S_n(B) \to \{0, 1\}$.*

What about continuous functions $f : S_n(B) \to C$, C some other comapct Hausdorff space? We will call this a C-valued formula over B. This leads to continuous logic. We will give examples next time.

Definition 1.8. Let C be a compact Hausdorff space (e.g. 2^ω, $[0, 1]$). By a *C-valued formula over B in variables $\bar{x} = (x_1, .., x_n)$*, we mean a continuous function $\phi : S_n(B) \to C$.

Given such a function ϕ and $\bar{a} \in \overline{M}^n$, we write $\phi(\bar{a}) = \phi(\mathrm{tp}(\bar{a}/B)) \in C$. Note that when $C = \{0, 1\}$, $\phi(\bar{a}) = 1$ iff $\models \varphi(\bar{a})$.

Example. Take $T = RCF$ in the language $(+, \times, 0, 1, -, <)$, i.e. $T = \mathrm{Th}(\mathbb{R}, +, \times, 0, 1, <)$. Let $\overline{M}$ be a big model. Let $X = [0, 1](\overline{M}) := \{x \in \overline{M} :\models 0 \leq x \leq 1\}$. Note that we may assume we have constants for all elements of the rationals (since the rationals are definable).

Define E on X to be the equivalence relation $E(x, y)$ iff $|x - y| < \frac{1}{n}$ for $n = 1, 2, 3, ...$ This is an ∞-definable equivalence relation over $\emptyset$ on X. Note that there is one class for each real number, i.e. continuum many classes. It is a bounded type-definable equivalence relation (i.e. there are boundedly many classes, and the bound is independent of the saturated model), with the number of classes less than the degree of saturation.

Let $C := X/E, \pi : X \to C$ with the "logic topology", i.e. $Z \subseteq X/E$ is closed iff $\pi^{-1}(Z)$ is defined by a possibly infinite collection of formulas over a small set of parameters.

Fact. C is a compact Hausdorff space under the logic topology.

Lemma 1.9.

1. *With the logic topology, C is homeomorphic to $[0, 1]$.*
2. *Fix a copy $\mathbb{R} \prec \overline{M}$. Then the map $X \to C \sim [0, 1]$ can be identified with the standard part map $st(-) : X \to [0, 1](\mathbb{R})$ (where for $a \in X$, $st(a) =$ the unique $r \in \mathbb{R}$ s.t. $|a - r| < \frac{1}{n}$ for all n).*
3. *Define $\pi' : S_1(T) \to C$ such that for $a \leq 0$, $\pi'(\mathrm{tp}(a)) = 0$; if $a \geq 1, \pi'(\mathrm{tp}(a)) = 1$; and if $0 \leq a \leq 1, \pi'(\mathrm{tp}(a)) = st(a)$. Then π' is continuous and surjective.*

[In fact, E induces a closed equivalence relation E' on $S_1(T)$, where $p \sim_E q$ if there is a realization a of p and b of q such that a and b are E-related. Then the usual topology on $[0, 1]$ is the quotient topology on $S_1(T)/E$.]

So you can recover the classical topology on the unit interval from "logic".

1.4 Imaginaries

Quotient objects are important in mathematics—in geometry, descriptive set theory, etc. (e.g. in algebraic geometry, GIT (geometric invariant theory), which is a theory of when the quotient of an algebraic variety by the action of an algebraic group is also a variety; in descriptive set theory, when is the quotient of a Polish space by a Borel equivalence relation a Polish space; etc.). We are interested in an account of quotient objects suitable for model theory.

Construction: Given $L, T, \overline{M}$, we construct "many-sorted" $L^{eq}, T^{eq}, \overline{M}^{eq}$ (by "many-sorted", we mean that there are several universes, or "sorts", and all function, variable, relation, and constant symbols are sorted. The quantifiers also come with sorts, so you can't use compactness in the usual way to "get out" of these sorts).

For each L-formula (L is one-sorted) $\phi(x_1, ..., x_n, y_1, ..., y_n)$ which defines an equivalence relation E on $\overline{M}^n$, introduce a new sort S_E (depending on E, not the defining formula). In particular, we have a sort $S_=$ for $x = y$.

We define L^{eq} as follows: For each such E, let f_E be a new function symbol from sort $S_=^n$ to S_E. For each n-ary relation symbol R in L, R will exist in L^{eq} as an n-ary relation symbol on $S_=$ (similarly for function symbols in L). We also introduce a separate equality symbol $=_{S_E}$ for each sort.

T^{eq} says that (1) each f_E is surjective; (2) given $\varphi(\bar{x}\bar{y})$ defining E, $\forall\bar{x}\forall\bar{y}(\varphi(\bar{x}, \bar{y}) \leftrightarrow f_E(\bar{x}) = f_E(\bar{y})$; (3) T is on $S_=$.

For any model M of T, M will have a unique expansion up to isomorphism to a model M^{eq} of T^{eq} (identifying M with $S_=(M^{eq})$ tautologically), and for E n-ary:

$$S_E(M^{eq}) := M^n/E(M)$$

Fact 1.10.

1. T^{eq} is complete. Any model M of T has a unique expansion M^{eq} to a model of T^{eq}. Moreover, $S_=$ has no more induced structure in T^{eq} than in T.

2. Let $\varphi(x_1, ..., x_k)$ be an L^{eq}-formula, where x_i is of sort S_{E_i}. Then there is an L-formula $\psi(\bar{y}_1, ..., \bar{y}_n)$ such that $T^{eq} \models \forall\bar{y}_1, ..., \bar{y}_n(\psi(\bar{y}_1, ..., \bar{y}_n) \leftrightarrow \varphi(f_{E_1}(\bar{y}_1), ..., f_{E_n}(\bar{y}_n)))$. Also, if σ an L-sentence, $T \models \sigma \iff T^{eq} \models \sigma$.

3. For $M, N \models T$, $M \preceq N \iff M^{eq} \preceq N^{eq}$.

4. $\overline{M}^{eq}$ is also $\overline{\kappa}$-saturated and strongly $\overline{k}$-homogeneous, with $|L^{eq}| = |L| + \omega$.

5. Any automorphism α of M exrends uniquely to an automorphism of M^{eq}. Note: identifying $M \models T$ with $S_=(M^{eq})$, $M^{eq} \subseteq \mathrm{dcl}(M)$.

6. The $\emptyset$-definable subsets of M^n in L coincide with the $\emptyset$-definable subsets of $S_=(M^{eq})^n$ in L^{eq}.

Definition. Let $\varphi(\bar{x}, \bar{y})$ be an L-formula. Define $E_\varphi(\bar{y}_1, \bar{y}_2)$ iff $\forall \bar{x}(\varphi(\bar{x}, \bar{y}_1) \leftrightarrow \varphi(\bar{x}, \bar{y}_2))$. Given $\bar{b}$, a *code* for the "definable set" $\varphi(\bar{x}, \bar{b})$ (in M or $\overline{M}$) is $\bar{b}/E_\varphi \in M^{eq}$ (resp. $\overline{M}^{eq}$). Note that $\alpha \in \mathrm{Aut}(\overline{M})$ will fix $\varphi(\bar{x}, \bar{b})(\overline{M})$ iff α fixes $\bar{b}/E_\varphi$.

So the family of sets defined by $\varphi(\bar{x}, \bar{b})$ as $\bar{b}$ varies is the set $\{\bar{b}/E_\varphi : \bar{b}\}$ which is $\emptyset$-definable in eq (this can be thought of as the moduli space for the family).

Definition 1.11. T has *elimination of imaginaries* if for any $a \in \overline{M}^{eq}$ if there exists $\bar{b} \in \overline{M}^n$ such that in $\overline{M}^{eq}$, $a \in \mathrm{dcl}(\bar{b})$ and $\bar{b} \in \mathrm{dcl}(a)$.

It has *weak E.I.* if $a \in \mathrm{dcl}(\bar{b})$ but $\bar{b} \in \mathrm{acl}(a)$. It has *geometric E.I.* if $a \in \mathrm{acl}(\bar{b})$ but $\bar{b} \in \mathrm{acl}(a)$

Remark. Assuming T has at least one constant symbol, T has E.I. iff in $\overline{M}$ (or any $M \models T$) for any $\emptyset$-definable equivalence relation E on $\overline{M}^n$ defined by $\varphi(\bar{x}, \bar{y})$, there is some k and a $\emptyset$-definable function (i.e. the graph is $\emptyset$-definable) such that $\overline{M} \models \forall \bar{x} \forall \bar{y}(\varphi(\bar{x}, \bar{y}) \leftrightarrow f(\bar{x}) = f(\bar{y}))$ (resp. M), so $f : \overline{M}^n \to \overline{M}^k$ induces a bijection between $\overline{M}^n/E$ and $\mathrm{Im}(f) \subseteq \overline{M}^k$, a $\emptyset$-definable set.

Exercise. The theory T of an infinite set in $L = \{=\}$ (which is complete and has QE) has weak E.I. but not E.I.. For consider $a \neq b$; then one can define an equivalence relation E on M in $L_=$ where $x \sim y$ iff either $x, y \in \{a, b\}$ or $x, y \in M \setminus \{a, b\}$. It follows that $\{a, b\} \in M^{eq}$. However, there is no tuple $\bar{c} \in M$ such that for any permutation α of M, α fixes $\{a, b\}$ iff α fixes $\bar{c}$ (since M is infinite, we can always find an α which fails this condition). Thus, by Fact 1.4, it follows that the code of $\{a, b\}$ in M^{eq} is not in $\mathrm{dcl}(\bar{c})$ for any $\bar{c}$, so T does not have E.I. On the other hand, the tuple $\bar{c} = ab$ and the code $d = \{a, b\}$ are such that $d \in \mathrm{dcl}(\bar{c})$ and $\bar{c} \in \mathrm{acl}(d)$, since, for instance, $f_E(a) = x \wedge f_E(b) = x$ defines d, while $\bar{c} = ab$ satisfies $f_E(x) = d \wedge f_E(y) = d$, which only has finitely many realizations and hence is a witness for $\bar{c} \in \mathrm{acl}(d)$.

Definition 1.12. 1. T has *definable (or built in) Skolem functions*, if for each $\varphi(\bar{x}, \bar{y})$, there is a partial $\emptyset$-definable function $f_\varphi(\bar{y})$ s.t. $T \models \forall \bar{y}(\exists \bar{x} \varphi(\bar{x}, \bar{y}) \to \varphi(f_\varphi(\bar{y}), \bar{y}))$, i.e. for some/any $M \models T$ and $\bar{b} \in M$, if $M \models \exists \bar{x}(\varphi(\bar{x}, \bar{b}))$ then $M \models \varphi(f_\varphi(\bar{b}), \bar{b})$.

2. T has *definable choice* if, moreover, $f_\varphi(\bar{b})$ depends only on the set $\varphi(\bar{x}, \bar{b})(M)$.

Remark 1.13. 1. Definable choice implies E.I. (but not the converse).

2. Existence of definable Skolem functions doesn't always imply E.I., e.g. $\mathrm{Th}(\mathbb{Q}_p, +, \times)$ has definable Skolem functions but not E.I.

3. In T^{eq}, definable choice is the same as having definable Skolem functions.

4. Examples:

   (a) RCF has definable choice (can be shown by induction on n for $\varphi(x_1, ..., x_n, y)$).

(b) If there is a model M of a complete theory encoding a total ordering, such that M has a well-ordering (not necessarily definable), then it has definable choice (by picking the least element of any definable set). By transfer, it holds for all M.

(c) Suppose T is strongly minimal (i.e. the universe is infinite, and definable with parameters subset of elements (not tuples) is finite or cofinite)) and $\mathrm{acl}(\emptyset)$ is infinite, then T has weak E.I. (also by induction on n). (see e.g. "Model theory of algebraically closed fields", Pillay, in Bouscaren volume)

5. (Skolemization) If T is countable, then $\exists$ some countable (complete) expansion T' of T in $L' \supseteq L$ with built-in Skolem functions (in fat, given by function symbols in L').

Chapter 2

ω-stability

2.1 ω-stability theory and Morley's theorem

Suppose T is countable, complete, in L, with no finite models. We have $\overline{M}$ as usual. Let's continue to work in a 1-sorted context (but everything works in T^{eq} many-sorted).

Definition 2.1. T is *ω-stable* if for any countable $A \subseteq \overline{M}$, $S_1(A)$ is countable. Equivalently, for any countable A, $S_n(A)$ is countable for all n (e.g. if we know this up to some n, we can show this for S_{n+1} by realizing the first coordinate of an $n+1$-type in a countable model; since A is countable, by induction it follows). Likewise for λ-stable for any $\lambda \geq \omega$.

Lemma 2.2. *$\forall \kappa > \omega$, there exists $M \models T$ of card κ such that for any countable $A \subseteq M$, only countable many complete n-types are realized in M (for any n).*

Proof. We may assume T is Skolemized via function symbols (since if the statement is true for a countable expansion of T, T countable, then it's true for T). Using Lemma 1.2 and 1.3 (existence and stretching of indiscernible sequences), we can find $M \models T$ with an infinite indiscernible (over $\emptyset$) sequence $B = (b_\alpha : \alpha < \kappa)$. So we may assume that M is the "Skolem hull" of the sequence B, i.e. $M = \{t(\bar{b}) : t \text{ an } L\text{-term}\}$, in which case $|M| = \kappa$.

Let $A \subseteq M$ be countable, $a \in A$ is of the form $t_a(\bar{b}_a)$ for some L-term t_a and tuple $\bar{b}_a$ from B. Let B_0 be the set of $b_\alpha, \alpha < \kappa$, appearing among these $\bar{b}_a$. So $B_0 \subseteq B$ countable. Using indiscernibility of B (and well-orderedness of κ), $\{\mathrm{tp}_M(\bar{b}/B_0) : \bar{b} \in B\}$ is countable (since the type of a tuple over B_0 depends only on the order type of the indices). As every element of M is of the form $t(\bar{b})$ for some $\bar{b} \in B$ and t an L-term (L countable), we get that there are countable many types realized in M for $n = 1$ (and by the same reasoning, for any tuple of length n). $\square$

Corollary 2.3. *If T is κ-categorical, then T is ω-stable.*

Proof. Suppose there is countable $A \subseteq \overline{M}$ s.t. $S_1(A)$ is uncountable. Then there is $M \prec \overline{M}$,

$|M| = \kappa$, such that uncountably many 1-types over A are realized in M. This + κ-categoricity contradicts the previous lemma. □

Definition 2.4. Fix n and a formula $\varphi(x_1, ..., x_n, \bar{a})$ (and let $\bar{x} = (x_1, ..., x_n)$. We define $\mathrm{RM}_n(\varphi(\bar{x}, \bar{a}))$ (the *Morley rank*) such that:

1. $\mathrm{RM}_n(\varphi(\bar{x}, \bar{a})) \geq 0$ if $\varphi(\bar{x}, \bar{a})$ is consistent.

2. $\mathrm{RM}_n(\varphi(\bar{x}, \bar{a})) \geq \delta$, δ limit, if $\mathrm{RM}_n(\varphi(\bar{x}, \bar{a})) \geq \alpha$ for all $\alpha < \delta$.

3. $\mathrm{RM}_n(\varphi(\bar{x}, \bar{a})) \geq \alpha+1$ if there are formulas $\psi_j(\bar{x}, \bar{b}_j)$, $j = 1, 2, 3, ...$, such that each $\psi_j(\bar{x}, \bar{b}_j)$ implies $\varphi(\bar{x}, \bar{a})$, and $\models \neg\exists\bar{x}(\psi_j(\bar{x}, \bar{b}_j) \wedge \psi_{j'}(\bar{x}, \bar{b}_{j'}))$ whenever $j \neq j'$, and $\mathrm{RM}_n(\psi_j(\bar{x}, \bar{b}_j)) \geq \alpha$.

We say that $\mathrm{RM}_n(\varphi(\bar{x}, \bar{a})) = \alpha$ if $\mathrm{RM}_n(\varphi(\bar{x}, \bar{a})) \geq \alpha$ but $\mathrm{RM}_n(\varphi(\bar{x}, \bar{a})) \not\geq \alpha + 1$. Otherwise, $\mathrm{RM}_n(\varphi(\bar{x}, \bar{a})) = \infty$.

This can also be done via "CB-rank" (Cantor-Bendixson rank) on $S_n(\overline{M})$ or $S_n(M')$ for M' ω-saturated.

Note that we can also define $\mathrm{RM}(p(x)) = \min\{\mathrm{RM}_n(\varphi) | \varphi \in p(x)\}$, for p a complete type.

Lemma 2.5. *Some observations:*

1. (a) *$RM_n(\varphi \vee \psi)$ = the max of the Morley ranks of φ and ψ.*

   (b) *$RM_n(\varphi) = 0$ if φ has only finitely many realization (i.e. φ algebraic)*

   (c) *Less obvious (but can be shown by considering RM in terms of CB-rank): suppose $RM_n(\varphi(\bar{x}, \bar{a})) = \alpha$. Then $\exists$ maximum d s.t. there are pairwise inconsistent $\psi_1(\bar{x}, \bar{a}_1), ..., \psi_d(\bar{x}, \bar{a}_d)$, each implying $\psi(\bar{x}, \bar{a})$ and each of $RM = \alpha$ (see e.g. Introduction to Stability). We call $d = dM(\varphi(\bar{x}, \bar{a}))$ (the Morley degree).*

Proposition 2.6. *TFAE:*

1. *T is ω-stable.*

2. *For all $\varphi(\bar{x}, \bar{a})$, $RM_n(\varphi(\bar{x}, \bar{a})) < \infty$.*

3. *T is λ-stable for all $\lambda \geq \omega$.*

Proof. 3 implies 1 is trivial.

For 1 implies 2: Suppose $\mathrm{RM}_n(\varphi(\bar{x}, \bar{a})) = \infty$. Then there are $\varphi_1(\bar{x}, \bar{a}_1), \varphi_2(\bar{x}, \bar{a}_2)$ both implying $\varphi_1(\bar{x}, \bar{a})$, mutually inconsistent, and each of RM $= \infty$. By iterating this, one obtains an infinite binary tree of formulas, such that at every level, every branch is consistent over countably many parameters. Thus, we obtain 2^{ω}-many types over countably many parameters, which contradicts ω-stability.

For 2 implies 3: Fix a model M of size λ. For each $p \in S_n(M)$, let $\varphi_p(\bar{x}, \bar{a}) \in p$ have least (RM, dM) (w.r.t the lexicographic ordering). Then the claim is that p is determined by φ_p: suppose $\varphi_p(\bar{x}) = \varphi_q(\bar{x})$, for RM $= \alpha$, dM $= d$. We first note that $\psi(\bar{x})$ over M is in $p(\bar{x})$ iff RM$(\neg\psi(\bar{x}) \wedge \varphi_p(\bar{x}))) < \alpha$: if RM$(\neg\psi(\bar{x}) \wedge \varphi_p(\bar{x}))) < \alpha$, then $\neg\psi(\bar{x}) \notin p$ (for otherwise, RM would be greater than or equal to α), so $\psi(\bar{x}) \in p$. Conversely, if $\psi \in p$, then RM$(\psi(x) \wedge \varphi_p(\bar{x})) = \alpha$ by Lemma 2.5. As $\psi(\bar{x}) \wedge \varphi_p(\bar{x}) \to \varphi_p(\bar{x})$, we have dM$(\psi(\bar{x}) \wedge \varphi_p(\bar{x})) \leq d$; but as the choice of α, d is minimal, it is equal to d. So if RM$(\neg\psi(\bar{x}) \wedge \varphi_p(\bar{x}))) \geq \alpha$, then RM$(\varphi_p(x)) > \alpha$, a contradiction. This is also true for q. Since RM$(\neg\varphi_q(x) \wedge \varphi_p(x)) < \alpha$ and vice versa, we have that $p = q$. □

A side remark: T is stable iff λ-stable whenever $\lambda = \lambda^\omega$; T is superstable iff λ-stable for all $\lambda \geq 2^\omega$.

Proposition 2.7 (Existence of saturated model)**.** *Assume T is ω-stable. Then for each cardinal κ and regular $\lambda \leq \kappa$, there is a λ-saturated model of T of card κ.*

Proof. Build a continuous elementary chain $(M_\alpha : \alpha < \lambda)$ (i.e. each M_α is the union of the models below it) of models of T of card κ such that all types in $S(M_\alpha)$ are realized in $M_{\alpha+1}$. By Proposition 2.6, we can do this as there exist $\leq \kappa$-many types in $S(M_\alpha)$, using downward Lowhenheim-Skolem to get the models to be of cardinality κ. Let $M = \bigcup M_\alpha, |M| = \kappa$. Then if $A \subseteq M$ of card $< \lambda$, then $A \subseteq M_\alpha$ for some $\alpha < \lambda$, by regularity of λ. By construction, all types over A are realized in $M_{\alpha+1}$, hence in M. (N.B.: can also do this if λ singular, but needs more). So M is λ-saturated. □

In particular, for each regular λ, there is a λ-saturated model of T of cardinality λ (i.e. by taking $\lambda = \kappa$).

Corollary 2.8. *Suppose T is κ-categorical, where $\kappa > \omega$. Then T has a κ-saturated model of size κ.*

Proof. By 2.3, T is ω-stable. By 2.7, it works if κ regular. So suppose κ is singular. In particular, κ is limit (as all successors are regular). So $\forall\lambda < \kappa$, λ^+ is regular and $< \kappa$. By 2.7 for all $\lambda < \kappa$, T has a model M_λ of size κ which is λ^+-saturated. By categoricity, it follows that the unique model of T of card κ is λ^+-saturated for all $\lambda < \kappa$. But then it follows that this model is κ-saturated. □

Now we turn to prime models.

Definition 2.9. Fix (small) $A \subseteq \overline{M}$. $M \prec \overline{M}$ is *prime over* A if $A \subseteq M$ and for any $A \subseteq N \prec \overline{M}$, there is an elementary embedding of $f : M \to N$ such that $f|_A = id$.

Proposition 2.10. *Assume T is ω-stable. Then $\forall A \subseteq \overline{M}$, there is a prime model M over A which is unique up to isomorphism over A (the isomorphism fixes A pointwise).*

Proof. We do existence first, and uniqueness will be done later. Let us remark that in the countable case, i.e. when $T_A = \mathrm{Th}(\overline{M}, a)$, $a \subseteq A$, is countable, then the general theory of countable models implies the existence of a prime model (by a back-and-forth argument). But we will continue with this next time. □

Discussion.

Recall we defined T is λ-stable for $\lambda \geq \omega$ and saw that ω-stable implies λ-stable for all $\lambda \geq \omega$.

Fact. T is λ-stable for some λ if and only if $\forall\lambda(T$ is λ-stable whenever $\lambda^\omega = lambda$, and T is super stable if T is λ-stable for all sufficiently large λ if and only if T is λ-stable for all $\lambda \geq 2^{\aleph_0}$.

ω-stable implies super stable implies stable.

Example. $\mathrm{Th}(\mathbb{Z}, +, 0)$ is superstable but not ω-stable. (QE after adding "$m \mid x$".)

Note. $\mathbb{Z} \supsetneq 2\mathbb{Z} \supsetneq 4\mathbb{Z} \supsetneq 2^n\mathbb{Z} \supsetneq$ ... finite index.

Example. $\mathrm{Th}(\mathbb{Z}^{(\omega)}, +)$ is stable not superstable. (Use the fact that $Th(G)$, where G is expansion of a group, is superstable, so there is no descending chain of definable subgroups $G \geq H_1 \geq H_2 \geq$ ... where each H_{n+1} has infinite index in H_n.)

In fact, if $(A, +)$ is any abelian group, then $Th(A, +)$ is stable. For abelian groups, ω-stable DCC is equivalent to no ∞ descending chain of definable subgroups, and superstable DCC is equivalent to no ∞ infinite index chain of definable subgroups.

End Discussion.

Recall that last time we defined a prime model $\overline{M} \models T$ over $A \subset \overline{M}$.

Definition 2.11. A model $M \prec \overline{M}$ containing A is said to be *constructible over* A if $M = A \cup \{b_\alpha : \alpha < \gamma\}$ such that $tp_{\overline{M}}(b_\beta / A \cup \{b_\alpha : \alpha < \beta\})$ is isolated (principal) for all $\beta < \gamma$.

(Recall: a type $p(x) \in S_1(B)$ is isolated if there is a formula $\varphi(x) \in p(x)$ such that $\overline{M} \models (\forall x)(\varphi(x) \to \psi(x)$ whenever $\psi(x) \in p(x)$.)

Remark 2.12.

1. If M is constructible over A then M is prime over A.
2. If M is constructible over A, then every finite tuple $\bar{b}$ in M, $tp(\bar{b}/A)$ is isolated (i.e. M is atomic over A).

Proof.

1. Let M be constructible over A and let $N \prec \overline{M}$ contain A.

   $\underline{\beta = 0}$: By constructible over A, $tp(b_0/A)$ is isolated by some formula $\varphi_0(x)$ with parameters in A. By elementary equivalence, there exists $a_0 \in N$ with $N \models \varphi_0(a)$, so $tp(a_0/A) = tp(b_0/A)$. Set $f_1 = id_A \cup \{(b_0, a_0)\}$.

   Assume now that for $\beta < \gamma$, we have $(a_\xi)_{\xi<\beta}$ and $f_\beta : A \cup \{b_\xi : \xi < \beta\} \to A \cup \{a_\xi : \xi < \beta\}$ with $f_\beta \restriction A = id_A$ and $f(b_\xi) = a_\xi$.

Inductive Step: For b_β, $tp(b_\beta/A \cup \{b_\alpha : \alpha < \beta\})$ is isolated by some formula $\varphi_\beta(x)$ with parameters in $A \cup \{b_\alpha : \alpha < \beta\}$. Find $a_\beta \in N$ with $N \models \varphi_\beta(a_\beta)$. If β is a successor, set $f_{\beta^+} = f_\beta \cup \{(b_\beta, a_\beta)\}$. Otherwise, set $f_{\beta^+} = \bigcup_{\alpha<\beta^+} f_\alpha \cup \{(b_\beta, a_\beta)\}$.

Let $f = \bigcup_{\beta<\gamma} f_\beta$; f elementarily embeds M into N.

2. Write $\bar{b} = (b_{i_0}, b_{i_1}, ..., b_{i_n})$, where $i_0 < i_1 < ... < i_n < \gamma$. We do induction on the maximal index i_n.

   $i_n = 0$: $\bar{b} = (b_0, ..., b_0)$. Since $tp(b_0/A)$ is isolated by some $\varphi(x)$ with parameters in A, $tp(\bar{b}/A)$ is isolated by $\theta(\bar{x}) = \bigwedge_{j\leq n} \varphi(x_j) \wedge \bigwedge_{j\leq n} x_0 = x_j$.

   Assume now that $tp(\bar{b}/A)$ is isolated for all $\bar{b} \in A \cup \{b_{\alpha:\alpha<\beta}$ for some $\beta < \gamma$.

   Inductive Step: Write $\bar{b}' = (b_{i_0}, b_{i_1}, ..., b_{i_{n-1}})$. By 1), $tp(b_{i_n}/A \cup \{b_i : i < i_n\})$ is isolated by some formula $\varphi(x)$ with finitely-many parameters in A and finitely-many parameters in $\{b_i : i < i_n\}$. Let $\bar{c}$ denote the parameters from $\{b_i : i < i_n\}$ and let $\bar{d} = \bar{b}'\bar{c}$. By Inductive Hypothesis, $tp(\bar{d}/A)$ is isolated by some formula $\psi(\bar{x}', \bar{y})$ with parameters in A and where $\bar{x}' = (x_0, x_1, ..., x_{n-1})$. Then, $\theta(\bar{x}) = (\exists\bar{y})(\psi(\bar{x}', \bar{y}) \wedge \psi(x_n, \bar{y})$ isolates $tp(\bar{b}/A)$.

□

Proposition 2.13. *If T is ω-stable, then over any (small) set $A \subset \overline{M}$, there is a constructible model.*

Proof. Let $A \subseteq \overline{M}$. If A is the universe of an elementary substructure $M \prec \overline{M}$, then we are done. Otherwise, by Tarski-Vaught, there exists $\varphi(x)$ over A such that $\overline{M} \models \exists x\varphi(x)$, but there is no $a \in A$ such that $\overline{M} \models \varphi(a)$. Choose such $\varphi(x)$ minimizing $(RM(\varphi), dM(\varphi))$.

Claim. $\varphi(x)$ isolates a complete 1-type over A.

Proof of Claim. Assume toward a contradiction that there is a $\psi(x)$ over A such that both $\varphi(x)\wedge\psi(x)$ and $\varphi(x)\wedge\neg\psi(x)$ are consistent. If both $\varphi(x)\wedge\psi(x)$ and $\varphi(x)\wedge\neg\psi(x)$ have $RM = \alpha$, then both of them have $dM < d$. Then, we have a contradiction, since each of them is a witness to a smaller RM and dM. So at least one of $\varphi(x) \wedge \psi(x)$ and $\varphi(x) \wedge \neg\psi(x)$ has $RM < \alpha$, which is also a contradiction since we chose $\varphi(x)$ minimizing (RM, dM).

Let $b_0 \models \varphi$ so $tp(b_0/A)$ is isolated. Continue with $A_1 = Ab_0$, taking unions at limit stages. Because we may choose b_α from a given $M \supseteq A$ and because the b_α's are distinct, we will eventually stop. □

Lemma 2.14. *Assume T is ω-stable. Let $M \models T$ with $|M| = \kappa > \omega$, $A \subseteq M$ with $|A| < \kappa$. Then, M contains an infinite (length ω) indiscernible over A sequence $(b_0, b_1, b_2, ...)$.*

Proof. Note $x = x$ has $> \lambda$-many realizations in M. Choose $\varphi(x)$ over M (φ L_M-formula) with $> \lambda$-many realizations in M and least $(RM(\varphi), dM(\varphi)) = (\alpha, d)$. We may assume φ is over A by adding to A.

Claim 1. $\varphi(x)$ determines (i.e. is in) a unique type $p_0(x) \in S_1(A)$ of $(RM, dM) = (\alpha, d)$. (N.B. Let $p(x)$ be a complete type over some set; $(RM(p), dM(p)) = \text{least}\{(RM(\psi), dM(\psi)) : \psi \in p\}$.)

Proof of Claim 1. We have to show $\{\varphi(x)\} \cup \{\neg\psi(x) : \psi(x) \to \varphi(x)\ (RM(\psi), dM(\psi)) < (\alpha, d)\}$ is consistent and determines a unique complete type. If not consistent, then we have $\overline{M} \models (\forall x)(\varphi(x) \to (\psi_1(x) \vee ... \vee \psi_k(x)))$ with $(RM(\psi_i), dM(\psi_i)) < (\alpha, d)$. Then, each $\psi_i(x)$ has $\leq \lambda$-many realizations in M, so $\varphi(x)$ also; contradiction. For uniqueness, assume t.a.c. that there is some $\theta(x)$ such that $\Phi(x) \cup \{\theta(x)\}$ and $\Phi \cup \{\neg\theta(x)\}$ are both consistent. Then, since $\theta(x) \wedge \varphi(x)$ and $\neg\theta(x) \wedge \varphi(x)$ both imply $\varphi(x)$, one of the two must have $(RM, dM) < (\alpha, d)$. WLOG, assume $(RM, dM)(\theta(x) \wedge \varphi(x)) < (\alpha, d)$. Then, $\neg\theta \vee \neg\varphi \in \Phi(x)$, but $\varphi(x) \in \Phi(x)$, so we conclude $\neg\theta(x)$ is implied by $\Phi(x)$, contradiction.

Claim 2. p_0 is realized in M.

Proof of Claim 2. There are $\leq \lambda$ formulas $\psi(x)$ over A of $(RM, dM) < (\alpha, d)$, each realized by at most λ-many people in M. So $\exists b_0 \in M$ realizing $\varphi(x)$ and none of the $\psi(x)$'s, i.e. b_0 realizes p_0. Continuing in this way to produce $b_0, b_1, b_2, ...$ in M. WMA A is infinite. Namely, given $b_0, ..., b_n$, let $A_n = A \cup \{b_0, .., b_n\}$. Let $p_{n+1}(x)$ be the unique complete type over A_n containing $\varphi(x)$ and with $(RM(\varphi), dM(\varphi))(\alpha, d)$. Let b_{n+1} realize p_{n+1}.

Claim 3. The sequence $(b_0, b_1, b_2, ...)$ is indiscernible over A.

Proof of Claim 3. By induction on n, we prove that if $i_0 < i_1 < ... < i_n$, then $\text{tp}(b_0, b_1, ..., b_n) = \text{tp}(b_{i_0}, b_{i_1}, ..., b_{i_n})$.

<u>$n = 0$</u>: Note b_{i_0} realizes $p_{i_0}(x)$ over $A_{i_0} \supseteq A$, which contains $\varphi(x)$ with $(RM(\varphi), dM(\varphi)) = (\alpha, d)$, so $p_{i_0} \supseteq p_0$. So $b_{i_0} \models p_0$.

<u>Inductive Step</u>: We have $i_0 < i_1 < ... < i_n$ and $\text{tp}(b_{i_n}/Ab_{i_0}...b_{i_{n-1}})$ has $(RM, dM) = (\alpha, d)$ and contains $\varphi(x)$. Observations:

1. $M \models \psi(b_n, b_0, ..., b_{n-1})$ if and only if $\varphi(x) \wedge \psi(x, b_0, ..., b_{n-1})$ has $(RM, dM) = (\alpha, d)$.

2. $M \models \psi(b_{i_n}, b_{i_0}, ..., b_{i_{n-1}})$ if and only if $\varphi(x) \wedge \psi(x, b_{i_0}, ..., b_{i_{n-1}})$ has $(RM, dM) = (\alpha, d)$. To see this, assume $M \models \psi(b_{i_n}, b_{i_0}, ..., b_{i_{n-1}})$. Then, $\psi(x, b_{i_0}, ..., b_{i_{n-1}}) \in \text{tp}(b_{i_n}/A \cup \{b_{i_0}, ..., b_{i_{n-1}}\} \subseteq p_{i_n}$. Since $\varphi(x) \wedge \psi(x, b_{i_0}, ..., b_{i_{n-1}}) \to \varphi(x)$, $(RM, dM)(\varphi(x) \wedge \psi(x, b_{i_0}, ..., b_{i_{n-1}}) \leq (\alpha, d)$. But $(RM, dM)(p_{i_n}) = (\alpha, d)$, so $(RM, dM)(\varphi(x) \wedge \psi(x, b_{i_0}, ..., b_{i_{n-1}}) = (\alpha, d)$. On the other hand, if $M \not\models \psi(b_{i_n}, b_{i_0}, ..., b_{i_{n-1}})$, then $M \models \neg\psi(b_{i_n}, b_{i_0}, ..., b_{i_{n-1}})$, so $(RM, dM)(\varphi(x) \wedge \neg\psi(b_{i_n}, b_{i_0}, ..., b_{i_{n-1}})) = (\alpha, d)$, so $(RM, dM)(\varphi(x) \wedge \psi(b_{i_n}, b_{i_0}, ..., b_{i_{n-1}})) \neq (\alpha, d)$.

By IH, $\text{tp}(b_0, ..., b_{n-1}/A) = \text{tp}(b_{i_0}, ..., b_{i_{n-1}}/A)$, so the RHS's of the above observations are equivalent, so the LHS's are too. □

Comment/Exercise. Suppose $\text{tp}(\bar{a}) = \text{tp}(\bar{b})$, then $(RM, dM)(\psi(x, \bar{a})) = (RM, dM)(\psi(x, \bar{b}))$.

Proposition 2.15. *Suppose T is ω-stable and for some $\kappa > \omega$ every model of T of size κ is κ-saturated. Then, for all $\lambda > \omega$ every model of T of cardinality λ is λ-saturated.*

[Our proof will allow us to replace the hypothesis by (*) "for some $\kappa > \omega$, every model of T of cardinailty κ is $\aleph_1$-saturated".]

Proof. Fix $\kappa > \omega$ as in the hypothesis. Let $\lambda > \omega$ be arbitrary. Suppose t.a.c. that there is a model $M \models T$ of size λ that is not λ-saturated. There is $A \subseteq M$ with $|A| < \lambda$ and $p(x) \in S_1(A)$ such that $p(x)$ is not realized in M. By Lemma 2.14, let $(a_i : i < \omega) \subset M$ be infinite and indiscernible over A. (*) Hence, there is no consistent formula $\psi(x)$ over $A \cup \{a_i : i < \omega\}$ such that $\psi(x) \to p(x)$ (i.e. such that for all $\varphi(x) \in p(x), M \models \forall x(\psi(x) \to \varphi(x))$. Otherwise, $\exists b \in M$ that realizes $\psi(x)$ and then b realizes $p(x)$). Now, a "downward" L-S argument allows us to assume A is countable.

Claim 1. There is a countable $A' \subseteq A$ such that no consistent formula over $A' \cup \{a_i : i < \omega\}$ implies $p'(x) = p \upharpoonright A'$.

Proof of Claim 1. (Extracting indiscernibles + downward L-S argument). Fix countable $A_0 \subseteq A$ (e.g. $A_0 = \emptyset$). For each consistent formula $\psi(x)$ over $A_0 \cup \{a_i : i < \omega\}$. By (*) above, there is a formula $\varphi_\psi(x) \in p(x)$ such that $M \models \exists x(\psi(x) \wedge \neg\varphi_\psi(x))$. Let $A_1 = A_0 \cup$ {the parameters from these $\varphi'_\psi s$}, so A_1 is countable. Continue to get $A_0 \subseteq A_1 \subseteq A_2 \subseteq ... \subseteq A$, all countable. Let $A' = \bigcup_{n<\omega} A_n \subseteq A$. Let $p'(x) = p \upharpoonright A'$. By construction, we see that there is no $\psi(x)$ over $A' \cup \{a_i : i < \omega\}$ implying $p'(x)$.

In Claim 1, we found a countable $A' \subseteq M$, $p'(x) \in S_1(A')$ and an A'-indiscernible sequence $(a_i : i < \omega) \subseteq M$ such that there for every consistent formula $\psi(x)$ over $A' \cup \{a_i : i < \omega\}$, there is $\varphi(x) \in p'(x)$ such that $\overline{M} \models \exists x(\psi(x) \wedge \neg\varphi(x))$.

We now return to κ and aim for a contradiction. By Remark 1.3, we can find $(b_i : i < \kappa) \subseteq \overline{M}$ indiscernible over A' with the same E-M type over A' as $(a_i : i < \omega)$. By Lemma 2.13, let $N \prec \overline{M}$ be a constructible model over $A' \cup \{b_i : i < \kappa\}$. Notice that $|N| = \kappa$ (as N is prime over $A' \cup \{b_i : i < \kappa\}$ and by DLS, there is an $N' \supseteq A' \cup \{b_i : i < \kappa\}$ which has cardinality κ.)

Claim 2. $p'(x) \in S_1(A')$ is not realized in N.

Proof of Claim 2. We know by Remark 2.12 that N is atomic (by constructible) over $A' \cup \{b_i : i < \kappa\}$. Suppose $p'(x)$ is realized in N by some c. Let $\mathrm{tp}(c/A' \cup \{b_i : i < \kappa\})$ is isolated by a formula $\varphi(x, b_{i_0}, ..., b_{i_n})$ where $i_0 < i_1 < ... < i_n$ and $\varphi(x, y_0, ..., y_n)$ is over A'. In particular, $N \models (\forall x)(\varphi(x, b_{i_0}, ..., b_{i_n}) \to \psi(x))$ for all $\psi(x) \in p' = \mathrm{tp}(c/A')$. But $\mathrm{tp}(a_0, ..., a_n/A') = \mathrm{tp}(b_{i_0}, ..., b_{i_n}/A')$ by choice of $(b_i : i < \kappa)$. Hence, $\models (\forall x)(\varphi(x, a_0, ..., a_n) \to \psi(x))$ for all $\psi(x) \in p'(x)$, contradicting Claim 1. So we seee that N is of cardinality κ and not $\aleph_1$-saturated. □

Corollary 2.16 (Morley's Theorem, enhanced.)**.** *T countable complete theory. TFAE:*

(1) T is κ-categorical for some $\kappa > \omega$.

(2) $\exists\kappa > \omega$ such that every model of T of card κ is $\aleph_1$-saturated.

(3) T is λ-categorical for all $\lambda > \omega$.

Proof. (1) $\implies$ (2). (1) implies T is ω-stable by Corollary 2.3. By Proposition 2.7, for any $\lambda > \omega$, T has a model of cardinality λ that is $\aleph_1$-saturated. SO for κ in (1), we have that every mode of T of cardinality κ is $\aleph_1$-saturated.

(2) $\implies$ (3). Assume (2).

Claim. T is ω-stable.

Proof of Claim. Otherwise, there is A countable and $|S_1(A)|$ uncountable. By Lemma 2.2, there is a model M of cardinality κ containing A that realizes only countably-many types over A. By (2), M is $\aleph_1$-saturated, but we have uncountably-many types in $S_1(A)$, so they could not all be realized in M. Contradiction.

So as T is ω-stable, by Proposition 2.15, we get T λ-categorical $\forall \lambda > \omega$. □

N.B. Just (1) $\iff$ (3) is easier. (1) $\implies$ ω-stability $\implies$ (3) by Proposition 2.15.

N.B.B. uncountable categoricity is not ABOUT cardinals. In fact, it is a theorem that T is uncountably categorical if and only if T is ω-stable and "unidimentional" (if p, q are complete types over M_1, M_2, then p is *nonorthogonal* to q.)

2.2 Stability theory

We want to bring in stability. Pourquoi: Let T be ω-stable. Define $\bar{a}$ independent of $\bar{b}$ over A iff $RM(\text{tp}(\bar{a}/A\bar{b})) = RM(\text{tp}(\bar{a}/A))$. What are its properties?

Let T be countably complete, and take a big $\overline{M} \models T$.

We proved Morley's theorem using only basic properties of ω-stable theories. To prove more sophisticated results, we need a theory of "independence." Stability theory provides such a theory, and it will help us understand ω-stable theories better.

Definition 2.17. Let $\varphi(x, y)$ be an L-formula.

1. Let $M \models T$. We say that $\varphi(x, y)$ is *stable in M* if there do not exist $a_i, b_i \in M$ for $i < \omega$ such that either, for all $i, j \in \omega$, $M \models \varphi(a_i, b_i)$ iff $i \leq j$ or, for all $i, j \in \omega$, $M \models \neg\varphi(a_i, b_i)$ iff $i \leq j$.

2. We say that $\varphi(x, y)$ is *stable (for T)* if $\varphi(x, y)$ is stable in M for every $M \models T$.

By compactness, the stability of a formula can be expressed in terms of our monster model.

Remark 2.18. A formula $\varphi(x, y)$ is stable for T iff there do not exist $a_i, b_i \in \overline{M}$ for $i < \omega$ such that $\overline{M} \models \varphi(a_i, b_j)$ iff $i \leq j$.

Why is this the case? Suppose that we have $a_i, b_i \in \overline{M}$ such that $\overline{M} \models \varphi(a_i, j_j)$ iff $i \leq j$. Then, by compactness, we also get $a'_i, b'_i \in \overline{M}$ such that $\overline{M} \models \neg\varphi(a'_i, b'_j)$ iff $i' \leq j'$.

Lemma 2.19. *1. If $\varphi(x,y)$ is stable (for T), then so is $\neg\varphi(x,y)$. If $\varphi(x,y)$ and $\psi(x,z)$ are stable (for T), then so are $(\varphi \vee \psi)(x,yz)$ and $(\varphi \wedge \psi)(x,yz)$.*

2. *If $\varphi(x,y)$ is stable (for T), then so is $\varphi^*(y,x)$, where $\varphi^*(y,x) \equiv \varphi(x,y)$.*

3. *A formula $\varphi(x,y)$ is stable (for T) iff $\varphi(x,y)$ is k-stable (for T) for some $k \in \omega$; i.e., for some k, it is not the case that there exist $a_i, b_i \in \overline{M}$ for $i \leq k$ such that $\varphi(a_i, j_i)$ holds iff $i \leq j$.*

Proof. Part (1) follows directly from the definition of stability of a formula.

Towards proving part (2), assume that $\varphi(x,y)$ is stable (for T). Suppose towards a contradiction that $\varphi^*(y,x)$ is not stable. By definition, it follows that there exists a model $M \models T$ and $a_i^*, b_i^* \in M$ for $i < \omega$ such that either, for all $i,j \in \omega$, $M \models \varphi^*(a_i^*, b_i^*)$ iff $i \leq j$ or, for all $i,j \in \omega$, $M \models \neg\varphi^*(a_i^*, b_i^*)$ iff $i \leq j$. Assume without loss of generality that the first of these two cases holds; the other case is analogous. Take $a_i := b_i^*$ and $b_i := a_{i+1}^*$. We have $M \models \neg\varphi(a_i, b_j)$ iff $M \models \neg\varphi^*(b_j, a_i)$ iff $M \models \neg\varphi(a_{j+1}^*, b_i^*)$ iff $j+1 \nleq i$ iff $i < j$.

It is immediate that, if $\varphi(x,y)$ is k-stable for some k, then $\varphi(x,y)$ is stable. Towards proving the other direction of part (3) (by contrapositive), assume that $\varphi(x,y)$ is not k-stable for any k. Note that the k-stability of $\varphi(x,y)$ is given by a sentence $\sigma_k \in T$ (which states that a certain bipartite graph omits the k-half graph). Let $\Sigma(x_0, x_1, \ldots, y_0, y_1, \ldots)$ be an infinite set of formulas asserting that, for each k, the finite sequences $x_0, \ldots, x_k$ and $y_0, \ldots, y_k$ together witness that $\varphi(x,y)$ is not k-stable. By our assumption, the set Σ is finitely consistent. By compactness, it follows that Σ is consistent, which means that we can find infinite sequences $a_0, a_1, \ldots$ and $b_0, b_1, \ldots$ witnessing that $\varphi(x,y)$ is not stable. □

Let G be the disjoint union of all (unipartite) finite graphs (V,E) up to isomorphism. Take $L = \{=, E\}$ and $M = G$. By compactness, the relation E is stable in M but not stable for $\mathrm{Th}(M)$.

Definition 2.20. 1. Fix a formula $\varphi(x,y)$ and $M \models T$. By a *φ-formula over M*, we mean a Boolean combination of instances $\varphi(x,a)$ of $\varphi(x,y)$ for $a \in M$.

2. By a *complete $\varphi(x,y)$-type over M*, we mean a maximal consistent collection of φ-formulas over M. Define $S_\varphi(M)$ to be the set of such complete φ-types over M.

3. For $a \in \overline{M}$, define $\mathrm{tp}_\varphi(a/M)$ to be the set of φ-formulas $\psi(x)$ over M true of a.

Remark 2.21. 1. The complete φ-types over M are precisely the collections of the form $\mathrm{tp}_\varphi(a/M)$ for $a \in \overline{M}$.

2. If $p(x) \in S_\varphi(M)$, then p is determined by which formulas $\varphi(x,b)$ or $\neg\varphi(x,b)$ are in p for each $b \in M$.

3. Define a *basic open* subset of $S_\varphi(M)$ to be one of the form $\{p(x) \in S_\varphi(M) : \psi(x) \in p\}$ for some φ-formula $\psi(x)$ over M. This basis turns $S_\varphi(M)$ into a compact, Hausdorff, totally disconnected space in which the clopen sets are given precisely by the φ-formulas..

Proof. The proof is left as an exercise and is similar to the proof of Definition/Fact 1.6. □

Note that any complete type $p(x) \in S_x(M)$ is a union of complete φ-types as $\varphi(x, y)$ varies; i.e., $p(x) = \bigcup_\varphi (p \restriction \varphi)$.

Definition 2.22. 1. Let $p(x) \in S_\varphi(M)$. We say that p is *definable* if there is a $\psi(y) \in L_M$ such that, for all $b \in M$, $\varphi(x, b) \in p(x)$ if and only if $M \models \psi(b)$. (Note that this $\psi(y)$ is unique up to logical equivalence.)

2. A complete type $p(x)$ is *definable* if, for each $\varphi(x, y) \in \mathcal{L}$, $p \restriction \varphi$ is definable.

We will soon see that T is stable (λ-stable for some $\lambda \geq \omega$) iff all complete types over all models $M \models T$ are definable. Nevertheless, there do exist unstable theories T and models of T over which all types are definable. For example, take $T = \mathrm{RCF}$ and $M = \mathbb{R}$, or $T = \mathrm{Th}(\mathbb{Q}_p)$ and $M = \mathbb{Q}_p$. Note that all types over M are definable iff, for all $\bar{b} \in \overline{M}$, all $\varphi(\bar{b}, \bar{y})$ where $\varphi \in \mathcal{L}$, and all $M \prec \overline{M}$, the set $\{\bar{a} \in M : \varphi(\bar{b}, \bar{a})\}$ is definable in M.

Theorem 2.23 (Fundamental theorem of local stability theory)**.** *Suppose $M \models T$ and $\varphi(x, y)$ is an L-formula which is stable in M. Let M^* be such that $M \prec M^*$ and M^* is $|M|^+$-saturated. Let $p(x) \in S_\varphi(M^*)$ be finitely satisfiable in M (i.e., such that each $\chi(x) \in p(x)$ is satisfied by some $a \in M$). Then $p(x)$ is definable, moreover by a φ^*-formula $\psi(y)$ over M (where φ^* is defined as in Lemma 2.19(2)). In fact, the converse is also true.*

For a model-theoretic proof, see the proof of Proposition 1.3.7 from *Topics in model theory* (Pillay 2024). To make life amusing, we give a proof that uses a special case of the following theorem from function theory from Grothendieck's thesis which appears in [4] and [15].

Theorem (Grothendieck 1952; see Pillay 2016)**.** *Let X be a compact Hausdorff space, and let $X_0 \subseteq X$ be a dense subset. Let A be a collection of continuous functions from X to $[0, 1]$ such that, if $f_i \in A$ and $x_i \in X_0$ for $i < \omega$, then $\lim_i \lim_j f_i(x_j) = \lim_j \lim_i f_i(x_j)$ whenever both sides of this equation exist. Let $f : X \to [0, 1]$ be in cl(A) with respect to the pointwise topology (i.e., with respect to the product topology on $[0, 1]^X$). Then f is continuous.*

To prove Theorem 2.23, we will use the special case of this theorem in which the $f \in A$ are $\{0, 1\}$-valued.

Proof of Theorem 2.23. Let $X := S_{\varphi^*}(M)$, and let $X_0 \subseteq X$ be the set of φ^*-types realized in M. For $a \in M$, let $f_a : X \to \{0, 1\}$ be given by

$$f_a(q) := \varphi(a, y)(q) = \begin{cases} 1 & \text{if } \varphi(a, y) \in q \\ 0 & \text{otherwise} \end{cases}.$$

For $q \in \mathrm{tp}_{\varphi^*}(b|M)$ for $q \in X_0$ (i.e., for $b \in M$), $\varphi(a, y)(q) = 1$ if $\models \varphi(a, b)$ and $\varphi(a, y)(q) = 0$ otherwise. Since φ is stable, the double limit condition from the statement of the theorem holds. (To see this, suppose towards a contradictiong that the double limit condition does not hold. A witness $\langle q_0, q_1, \dots \rangle$ to this would (after trimming off a possibly empty initial segment) also witness that $\varphi(x, y)$ has the order property , contradicting the assumption that $\varphi(x, y)$ is stable.)

Now, suppose that $p(x) \in S_\varphi(M^*)$ is finitely satisfiable in M. Then, for any $b \in M^*$, whether or not $\varphi(x, b) \in p$ depends only on $q_* = \mathrm{tp}_{\varphi^*}(b/M)$. *Pourquoi? Suppose that $b_1, b_2 \in M^*$ have the same φ^*-type over M but $\varphi(x, b_1) \in p$ and $\neg\varphi(x, b_2) \in p$. Then $\varphi(x, b_1) \wedge \neg\varphi(x, b_2) \in p$. It follows that there is an $a \in M$ such that $\models \varphi(a, b_1) \wedge \neg\varphi(a, b_2)$, which is a contradiction.* Thus, it follows that p determines a function $f : X = S_{\varphi^*}(M) \to \{0, 1\}$.

Claim. We have $f \in \mathrm{cl}(A)$.

Proof of Claim. Let $q_1, \dots, q_n \in X$ be realized by $b_1, \dots, b_n \in M^*$. Since $p(x)$ is finitely satisfiable in M, there exists $a \in M$ such that, for each $i \in \{1, \dots, n\}$, $\varphi(x, b_i) \in p$ if and only if $\models \varphi(a, b_i)$. Therefore, $f(q_i) = f_a(q_i)$ for $i \in \{1, \dots, n\}$. It follows that $f \in \mathrm{cl}(A)$. □

It follows from the theorem that $f : S_{\varphi^*}(M) \to \{0, 1\}$ is continuous. Therefore, there is a φ^*-formula $\psi(y)$ over M such that $f(q) = 1$ if and only if $\psi(y) \in q$. This property translates into the following: for all $b \in M^*$, $\varphi(x, b) \in p(x)$ if and only if $\models \psi(b)$. □

Corollary 2.24. *Suppose that $\varphi(x, y)$ is stable in M. Then any $p(x) \in S_\varphi(M)$ is definable (by a φ^*-formula) over M.*

Proof. Take $M^* \succ M$ sufficiently saturated. Given p, let $p^* \in S_\varphi(M^*)$ such that $p(x) \subseteq p^*(x)$ and such that p^* is finitely satisfiable in M. Apply Theorem 2.23 to p^* to get that p is definable by some ψ. □

Corollary 2.25. *Given $\varphi(x, y) \in \mathcal{L}$, the following are equivalent:*

1. *The formula φ is stable (for T).*
2. *For all $M \models T$, and $p(x) \in S_\varphi(M)$, $p(x)$ is definable.*
3. *For all $\lambda \geq \omega$ and $M \models T$ of cardinality λ, $|S_\varphi(M)| \leq \lambda$.*
4. *There exists a $\lambda \geq \omega$ such that, for all $M \models T$ of cardinality λ, $|S_\varphi(M)| \leq \lambda$.*

Proof. The implication (1) → (2) is given by Corollary 2.24. The implication (2) → (3) is proven by counting the number of formulas defining each φ-type. (Any φ-type $p(x) \in S_\varphi(M)$ is determined by its definition $\psi(y)$ over M.) The implication (3) → (4) is obvious.

We will prove the implication (4) → (1) by contrapositive. Assume that $\varphi(x, y)$ is unstable (for T), and fix a $\lambda \geq \omega$. We can find a total ordering I of cardinality λ with strictly more than λ many initial segments J. (To do so, let μ be the least cardinal such that $2^\mu > \lambda$, and take I be the set of eventually constant functions from $\mu \to 2$, ordered lexicographically. One can check that this I has the desired condition.) By compactness and the instability of φ, there exist a_J for each initial segment J of I and $b_i \in \overline{M}$ for each $i \in I$ such that $\models \varphi(a_J, b_i)$ if and only if $i \in J$. Let $M \prec \overline{M}$ have cardinality λ and contain all the b_i's. Then the a_J's all realize different φ types over M, which implies that $|S_\varphi(M)| > \lambda$. □

Recall that we defined T to be λ-stable iff $|S(M)| \leq \lambda$ for all models $M \models T$ of cardinality λ.

Corollary 2.26. *Given a countable complete theory T, the following are equivalent:*

1. *All $\varphi(x, y) \in \mathcal{L}$ are stable (for T).*
2. *For any model $M \models T$, each $p(x) \in S(M)$ is definable.*
3. *The theory T is λ-stable whenever $\lambda = \lambda^{\aleph_0}$.*
4. *The theory T is λ-stable for some $\lambda \geq \omega$.*

Proof. Exercise. □

Shelah's proof that (1) ⇒ (2) is elementary and uses 2-rank. However, Theorem 2.23 says much more than Corollary 2.26, and we will use its full strength below to deduce the following symmetry statement.

Proposition 2.27. *Suppose $\varphi(x, y)$ is stable in M, and let $p(x) \in S_\varphi(M)$ and $q(y) \in S_{\varphi^*}(M)$. Let $\psi(y)$ be the φ^*-formula over M defining of $p(x)$ (coming from Theorem 2.23). Similarly, let $\chi(x)$ be the φ^*-formula over M defining $q(y)$. We have that $\psi(y) \in q$ if and only if $\chi(x) \in p$.*

Proof. Let $M \prec M^*$, where M^* is $|M|^+$-saturated. Let $p^* \in S_\varphi(M^*)$ with $p \subseteq p^*$ and p^* finitely satisfiable in M. By Theorem 2.23, we have that $\psi(y)$ is the (φ)-definition of p^*. Suppose that $\psi(y) \in q(y)$, and let b realize q in M^*. It follows that $\varphi(x, b) \in p^*$. We will prove that $\chi(x) \in p$, the other part of the proof being analogous. Suppose towards a contradiction that $\neg\chi(x) \in p \subseteq p^*$. It follows that $\neg\chi(x) \wedge \varphi(x, b) \in p^*$. As p^* is finitely satisfiable in M, we have $\models \neg\chi(a) \wedge \varphi(a, b)$ for some $a \in M$. It follows that $\varphi(a, y) \in q(y)$ and we get a contradiction to $\chi(x)$ being the definition of q. □

Little remark. Take $p(x) \in S_\varphi(M)$ definable by $\psi(y)$ over M. Then, for all $M' \succ M$, define $p'(x) := \{\varphi(x, b) : b \in M' \text{ and } \models \psi(b)\} \cup \{\neg\varphi(x, b) : b \in M' \text{ and } \models \neg\psi(b)\}$. Then $p'(x)$ is consistent, because

$$M \models \forall y_1, \dots, y_n \forall z_1, \dots, z_n (\psi(y_1) \wedge \dots \wedge \psi(y) \wedge \neg\psi(z_1) \wedge \dots \psi(z_n) \to \exists x(\varphi(x, y_1) \wedge \dots \wedge \varphi(x, y_n) \wedge \neg\varphi(x, z_1) \wedge \dots \wedge \neg\varphi(x, z_n)).$$

Thus, there is a unique $p'(x) \in S_\varphi(M')$ such that $p \subseteq p'$. So, Theorem 2.23 can be restated as follows. Let $\varphi(x, y)$ be stable and $p(x) \in S_\varphi(M)$. Then, for any $M' \succ M$, the unique definable extension of p over M' coincides with the unique $p' \in S_\varphi(M')$ which is finitely satisfiable in M.

Now, let us pass back to ω-stable theories T. By Corollary 2.26, every $\varphi(x, y)$ is stable for T. We also have access to Theorem 2.23. We want to understand the "interpretation" of independence (where $\overline{a}$ is independent from $\overline{b}$ if $\mathrm{RM}(\mathrm{tp}(\overline{a}/A)) = \mathrm{RM}(\mathrm{tp}(\overline{a}/A, \overline{b}))$).

In what follows, fix some ω-stable theory T.

Exercise 2.28.

1. Suppose that $p(x) \in S_x(A)$, and suppose that $\mathrm{RM}(p) = \alpha$ and $\mathrm{dM}(p) = 1$. Then, for each $B \supseteq A$, $p(x)$ has a unique extension to a type $q(x) \in S_x(B)$ with $\mathrm{RM}(q) = \alpha$.
2. More generally, suppose that $p(x) \in S_x(A)$, and suppose that $\mathrm{RM}(p) = \alpha$ and $\mathrm{dM}(p) = d$. Then, for each $B \supseteq A$, there are some, and at most d many extensions $p(x) \subseteq q(x) \in S_x(B)$ with $\mathrm{RM}(q) = \alpha$.

Remark 2.29. Suppose that M is ω-saturated and that $p(x) \in S_x(M)$. Then $\mathrm{dM}(p) = 1$.

Proof. Let $\theta(x) \in p(x)$ have Morley rank at least α. Suppose that $\mathrm{dM}(\theta) = k$. It follows that there are $b_1 \in \overline{M}$ and formulas $\psi_1(x, b_1), \ldots, \psi_k(x, b_k)$ such that, $\mathrm{RM}(\psi_i(x, b_i)) = \alpha$ and $\mathrm{dM}(\psi_i(x, b_i)) = 1$ for each i and $\overline{M} \models \forall x(\theta(x) \leftrightarrow \psi_1(x, b_1) \vee \cdots \vee \psi_k(x, b_k))$. By ω-saturation of M, there exist $b'_1, \ldots, b'_k \in M$ with the same type over parameters in θ as $b_1, \ldots, b_k$. Therefore, $\mathrm{RM}(\psi_i(x, b'_i)) = \alpha$ and $\mathrm{dM}(\psi_i(x, b'_i)) = 1$ for each i. As $p(x)$ is complete, we have $\psi(x, b_i) \in p$ for some i. ☐

Lemma 2.30. *Let $p(x) \in S_x(A)$ and let $M \supseteq A$ be sufficiently saturated/homogeneous. Let $p'(x) \in S_x(M)$ be such that $p(x) \subseteq p'(x)$ and $RM(p') = RM(p) = \alpha$. By Remark 2.29, we may assume that $dM(p') = 1$. If T is stable, p is definable over $acl^{eq}(A)$. (I.e., for all $\varphi(x, y) \in L$, the φ-definition $\psi(y)$ of φ over is equivalent to a formula $\psi'(y)$ with parameters from $acl^{eq}(A)$. As described above, this means that $\psi(y)$ is almost over A (working over M^{eq}).)*

Proof. Let $\sigma \in \mathrm{Aut}(M/A)$. So, $\sigma(p') \supseteq p(x)$ and $\mathrm{RM}(\sigma(p')) = \alpha$. By Exercise 2.28(2), there exist finitely many possibilities for $\sigma(p')$. Fix $\varphi(x, y)$, and let $\psi(y)$ be the φ-definition of $p'(x)$. Therefore, there exist finitely many images for $\sigma(\psi(y))$ of $\psi(y)$ (up to logical equivalence) for $\alpha \in \mathrm{Aut}(M/A)$. By Fact 1.4, $\psi(y)$ is almost over A. (In fact, let $\ulcorner\psi(y)\urcorner$ be the canonical parameter of $\psi(y)$ in M^{eq}. Then $\ulcorner\psi(y)\urcorner \in \mathrm{acl}^{\mathrm{eq}}(A)$.) ☐

Proposition 2.31. *Let M be sufficiently saturated/homogeneous, $A \subset M$ small, $p(x) \in S_x(M)$ definable over A. Let $p(x)|_A$ = set of formulas in p which are over A. Then $(RM, dM)(p|_A) = (RM, dM)(p) = (\alpha, 1)$.*

Proof. Let $\phi(x, c) \in p$ have $RM = \alpha$, $dM = 1$, $q(y) = tp(c/acl^{eq}(A))$, $RM(q) = \beta$. Let $q'(y) \in S_y(M)$ extend $q(y)$ with $RM = \beta$ (cf. exercise 2.28). By lemma 2.30, let $\psi(x)$ be the ϕ^*-definition of $q'(y)$ (i.e. for $a \in M$, $\models \psi(a)$ iff $\phi(a, y) \in q'(y)$.) So $\psi(x)$ is over $acl^{eq}(A)$.

Main claim: $\psi(x) \sim_{RM} \phi(x,c)$ i.e. $RM(\psi(x)\Delta\phi(x,c)) < \alpha$.

This will imply $\psi(x) \in p(x)$ and $\psi(x)$ has RM, dM $(\alpha, 1)$. Then noting that $p(x)$ is definable over A, each of the finitely many A-conjugates $\psi(x)$ will also have $(RM, dM) = (\alpha, 1)$ and all be in p. So $\bigvee_{i=1}^{n} \psi(x)$ has $(RM, dM) = (\alpha, 1)$ and is in p.

Proof of main claim. Part (a): $RM(\phi(x,c) \wedge \neg\psi(x)) < \alpha$. Otherwise $RM(\phi(x,c) \wedge \neg\psi(x)) = \alpha$, so $\neg\psi(x) \in p$. Let $M \prec M^*$ where M^* is $|M|^+$-saturated, and let $p^*(x)$ be the definable extension of p over M^*. Let c' realize $q'(y)$ in M, so c and c' have the same type over A. As $\phi(x,c) \in p \subseteq p*$, $\phi(x,c') \in p*$, so $\phi(x,c') \wedge \neg\psi(x) \in p*$. By 2.23, there is an $a \in M$ s.t. $\models \phi(a,c') \wedge \neg\psi(a)$, contradicting that $\psi(x)$ is the ϕ^*-definition of $q'(y)$.

Part (b): $RM(\psi(x) \wedge \neg\phi(x,c)) < \alpha$. Otherwise, $RM(\psi(x) \wedge \neg\phi(x,c)) = \gamma \geq \alpha$. As $RM(\phi(x,c)) = \alpha$ it follows that $RM(\psi(x)) = \gamma$. Let $M \prec M^*$ as before and let $r(x) \in S_x(M)$ contain $\psi(x)\wedge\neg\phi(x,c)$ with $RM(r) = \gamma$. Let $r^*(x) \in S_x(M^*)$ be an extension of r with $RM = \gamma$. As $\psi(x) \in r^*|_A$ has $RM = \gamma$, r^* is definable over $acl^{eq}(A)$ by 2.30. Let $c' \in M^*$ realize $q'(y)$ as before. Then since c and c' have the same type over $acl^{eq}(A)$ and $\psi(x) \wedge \neg\phi(x,c) \in r \subseteq r^*$, $\psi(x) \wedge \neg\phi(x,c') \in r^*$. Again by 2.23, there is $a \in M$ s.t. $\models \psi(a) \wedge \neg\phi(a,c)$, contradicting that $\psi(x)$ is the ϕ^*-definition of $q'(y)$. □

□

Corollary 2.32. *Let $p(x) \in S_x(M)$, $M \prec \overline{M}$. Then $dM(p) = 1$.*

Proof. Let $M \prec M^*$ sufficiently saturated and let $p'(x) \in S_x(M^*)$, $p \subseteq p^*$, $RM(p) = RM(p') = \alpha$. Then by lemma 2.30, p' is definable over M, so by proposition 2.31, $(RM, dM)(p) = (RM, dM)(p') = (\alpha, 1)$. □

(Nb: the same proof shows that any complete type $p(x)$ over $acl^{eq}(A)$ has $dM = 1$.)

Definition 2.33.

1. In $\overline{M}$ we say $\overline{a}$ is *independent* from B over A (where $A \subseteq B$) if $RM(tp(\overline{a}/B)) = RM(tp(\overline{a}/A))$ (We may also say $tp(\overline{a}/B)$ is a *nonforking extension* of $tp(\overline{a}/A)$).
2. We say $tp(\overline{a}/A)$ is *stationary* if it has a unique nonforking extension over any $B \supseteq A$ (equivalently, for every $M \supseteq A$, $tp(\overline{a}/A)$ has a unique nonforking extension over M).

Proposition 2.34.

1. *Let $q(x) = tp(\overline{a}/B)$ and $A \subseteq B$. Then $\overline{a}$ is independent from B over A iff there is a complete extension $q'(x)$ of $q(x)$ over some $M \supseteq B$ s.t. $q'(x)$ is definable over $acl^{eq}(A)$.*
2. *$p(x) \in S_x(A)$ is stationary iff $dM = 1$.*

Proof. 1: Suppose $\overline{a}$ is *independent* from B over A, so $RM(q) = RM(tp(\overline{a}/A)) = \alpha$. Let $M \supseteq B$. By 2.28 let $q'(x) \in S_x(M)$ with $RM(q') = \alpha$, $q' \supseteq q$. By 2.30 $q'(x)$ is definable over $acl^{eq}(A)$. Conversely, suppose $q'(x) \in S_x(M)$ extends q and is definable over $acl^{eq}(A)$. We may assume M is suitably saturated. By 2.31, $RM(q'|_{acl^{eq}(A)}) = RM(q') = \beta$, so there is a formula $\psi(x) \in q'|_{acl^{eq}(A)}$ with $RM(\psi) = \beta$. Then the disjunction of the A-conjugates of ψ has $RM = \beta$ and is in $q|_A = tp(\overline{a}/A)$, so $q = tp(\overline{a}/B)$ and $tp(\overline{a}/A)$ have the same Morley Rank, so $\overline{a}$ is *independent* from B over A.

2: Suppose $p(x) \in S_x(A)$ has $(RM, dM) = (\alpha, 1)$. Then for any $B \supseteq A$, p has a unique extension $p'(x) \in S_x(B)$ with the same RM by 2.28. Conversely, suppose $RM(p) = \alpha$ and $dM(p) = d > 1$. Choose $\phi(x) \in p$ with $(RM, dM) = (\alpha, d)$. Then there are $\phi_1(x), \ldots, \phi_d(x)$ (over additional parameters) mutually disjoint, each with $(RM, dM) = (\alpha, 1)$ such that $\phi(x) \leftrightarrow \bigvee_{i=1}^{d} \phi_i(x)$. Let $M \supseteq A$ contain the parameters of the $\phi_i's$. Let $p_i(x) \in S_x(M)$ contain $\phi_i(x)$ and have $RM = \alpha$. Then the $p_i's$ are distinct types over M each extending p with $RM = \alpha$, so p is not stationary. ☐

Proposition 2.35. *For $A \subseteq B \subseteq C$,*

1. *Transitivity: if $\overline{a}$ is independent from B over A and $\overline{a}$ is independent from C over B then $\overline{a}$ is independent from C over A*

2. *Existence: Given $p(\overline{x}) \in S(A)$ and $B \supseteq A$ there is a nonforking extension $q(\overline{x}) \in S(B)$ of $p(\overline{x})$.*

3. *Local character: $\overline{a}$ is independent from B over A iff $\overline{a}$ is independent from $A\overline{b}$ over A for all finite tuples $\overline{b} \in B$.*

4. *Given $q(x) \in S(M)$ there exists a finite $A \subseteq M$ s.t. $q(x)$ does not fork over A and $q|_A$ is stationary.*

5. *Symmetry: $\overline{a}$ is independent from $A\overline{b}$ over A iff $\overline{b}$ is independent from $A\overline{a}$ over A.*

Proof. 1. Immediate

2. From definitions, plus 2.28

3. Because RM depends on formulas.

4. $q(x)$ has $(RM, dM) = (\alpha, 1)$ witnessed by $\phi(x, \overline{c}) \in q$. Then $q|_{\overline{c}}$ has $(RM, dM) = (\alpha, 1)$, so $\overline{c} \subseteq M$ works.

5. Fix $\overline{a}, \overline{b}, A$. We may assume $A = acl^{eq}(A)$ as all relevant Morley ranks are unchanged. Let $M \supseteq A$ be a sufficiently saturated model containing $\overline{a}, \overline{b}$. Let $\overline{a'}$ realize $\text{tp}(\overline{a}/A)$ with $\overline{a}'$ independent of M over A, and likewise for $\overline{b}'$. So $\text{tp}(\overline{a}'/M)$ and $\text{tp}(\overline{b}'/M)$ are each definable over A by lemma 2.30. So $\text{tp}(\overline{a}'/A)$, $\text{tp}(\overline{a}/A\overline{b})$, and $\text{tp}(\overline{a}'/M)$ all have the same $RM = \alpha$ and $dM = 1$, by 2.31 and the fact that $\text{tp}(\overline{a}/A\overline{b}) = \text{tp}(\overline{a}'/A\overline{b})$. Suppose $\models \phi(\overline{a}, \overline{b})$. Then

$\models \phi(\overline{a}', \overline{b})$ because $tp(\overline{a}/A\overline{b}) = tp(\overline{a}'/A\overline{b})$. So if $\psi(\overline{y})$ is the ϕ-def of $tp(\overline{a}'/M)$ over A, then $\models \psi(\overline{b})$ so $\models \psi(\overline{b}')$, so letting $\chi(\overline{x})$ be the ϕ^*-def of $tp(\overline{b}'/M)$, by 2.27 we have $\models \chi(\overline{a}')$, so $\models \chi(\overline{a})$, so $\models \phi(\overline{a}, \overline{b}')$. Thus $tp(\overline{b}/A\overline{a}) = tp(\overline{b}'/A\overline{a})$, so since $\overline{b}'$ is *independent* from $A\overline{a}$ over A, $\overline{b}$ is *independent* from $A\overline{a}$ over A.

□

2.3 Canonical Bases

Suppose T stable, $M \models T$ sufficiently saturated/homogeneous. Any $p(x) \in S_x(M)$ is definable i.e. for each $\phi(x, y) \in L$ there is $\psi_\phi(y)$ over M s.t. $\phi(x, y) \in p$ iff $\models \psi_\phi(b)$ for all $b \in M$. We define $Cb(p) = \mathrm{dcl}^{eq}(\ulcorner \psi_\phi \urcorner : \phi \in L)$.

Proposition 2.36. *If T is ω-stable, $p(x) \in S_x(M)$, M sufficiently saturated, then $Cb(p)$ is the definable closure (in eq) of a finite set.*

Proof. Let $A = Cb(p)$ (note that A is countable). Then $p(x)$ is definable over A since any formula is definable over its code. So by 2.31 there is $\chi(x)$ over A with $(RM, dM)(\chi(x)) = (\alpha, 1) = (RM, dM)(p)$. Let d be the parameter from A in the formula $\chi(x)$. Then for $\alpha \in Aut(M)$, $\alpha(p) = p$ iff $\alpha(d) = d$. So as $d \in A$, α fixes d, so it fixes a formula with $RM = \alpha$, $dM = 1$ in p so it fixes p. So $A = dcl^{eq}(d)$. □

Interlude: let $\chi(x, d)$ be the formula given above with $d \in A$, $\chi(x, d) \in p$ of $RM = \alpha$, $dM = 1$. Let $tp(d') = tp(d)$, then $\chi(x, d') \sim_{RM} \chi(x, d)$ iff $\chi(x, d') \leftrightarrow \chi(x, d)$.

2.4 Morley Sequences

Definition 2.37. Let $p(x) \in S_x(A)$ be stationary. By a Morley sequence in $p(x)$ of length κ we mean $(b_\alpha : \alpha < \kappa)$ where b_0 realizes p and b_α realizes the unique nonforking extnesion of p over $A \cup \{b_\beta : \beta < \alpha\}$.

Lemma 2.38. *Any Morley sequence in $p(x)$ over A is indiscernible over A. Moreover, it is totally indiscernible–i.e., for all distinct $\alpha_1, \ldots, \alpha_n < \kappa, \beta_1, \ldots, \beta_n < \kappa$, $tp(b_{\alpha_1}, \ldots, b_{\alpha_n}/A) = tp(b_{\beta_1}, \ldots, b_{\beta_n}/A)$.*

Proof. First part: as in proof of 2.14. Second part: let $f : \kappa \to \kappa$ be a bijection. By forking calculus $b_{f(\alpha)} : \alpha < \kappa$ is also a Morley sequence in p. □

Remark 2.39. In fact, if T is stable then any indiscernible sequence over any set is totally indiscernible (Pillay 1983, prop 71).

Definition 2.40. We say $\{b_i : i \in I\}$ is A-independent (independent over A) if for each $i \in I$ b_i is independent from $A \cup \{b_j : j \neq i \in I\}$ over A.

Note: A Morley sequence is an indepndent set (over the base).

Lemma 2.41. *Suppose* $\{b_i : i \in I\}$ *is* A*-independent and there is a finite tuple* $\bar{c}$ *dependent on* b_iA *over* A *(*$\bar{c}$ *forks with* b_i *over* A*) for all* $i \in I$*. Then* I *is finite.*

Proof. Suppose not, so $I = \omega$. Then by forking calculus, we show that $\bar{c}$ dependent on $b_0, \ldots, b_i, b_{i+1}A$ over $b_0, \ldots, b_iA$ for all $i \in I$. Then $RM(tp(\bar{c}/A)) > RM(tp(\bar{c}/Ab_0)) > RM(tp(\bar{c}/Ab_0, \ldots, b_i)) > \ldots$, a contradiction since Morley ranks are ordinals. □

(Cf. lemma 2.245 of AMS book)

2.5 Saturated Models Revisited

Lemma 2.42. *Let* $\lambda \geq \omega$*, then* T *has a* λ*-saturated model of cardinality* λ*.*

Proof. By λ-stability of T we construct an elementary chain $M_\alpha : \alpha < \lambda$ of models of T of cardinality λ s.t. all types in $S(M_\alpha)$ are realized in $M_{\alpha+1}$. Let $M = \bigcup_{\alpha<\lambda} M_\alpha$, so $|M| = \lambda$. Claim: $|M|$ is λ-saturated.

Proof: let $A \subseteq M$, $|A| < \lambda$, $p(x) \in S(A)$. When $\lambda = \omega$ A is λ-saturated because it is finite. Otherwise we may assume by downwards Löwenheim–Skolem that $A = N \prec M$ of size $< \lambda$, $p(x) \in S(N)$. Let $A_0 \subseteq N$ be such that $p(x)$ is definable over A_0 by 2.31. Then $A_0 \subseteq M_\alpha$ for some α, so say $A_0 \subseteq M_0$. By construction of M we can find in M a Morley sequence $(b_\alpha : \alpha < \lambda)$ in $p|_{A_0}$. For each finite $\bar{c} \in N$, by 2.41 $\bar{c}$ depends on at most finitely many b_α over A_0, so by symmetry at most finitely many b_α depend on $\bar{c}$ over A_0. So by the finite character of independence there are at most $|N|$-many b_α which depend on N over A_0. But $|N| < \lambda$, so there exists some $\alpha < \lambda$ s.t. b_α is independent from N over A_0. So b_α realizes the unique nonforking extension of $p|_{A_0}$ to N which is p. □

Lemma 2.43. *Given* $\phi(x, y) \in L$*, there exists* $k < \omega$ *s.t. for any infinite indiscernible (and so totally indiscernible by remark 2.39) sequence* $(a_i : i \in I)$ *and any* $b \in \overline{M}$ *either there are at most* k a_i'*s s.t.* $\models \phi(a_i, b)$ *or at most* k a_i'*s s.t.* $\models \neg\phi(a_i, b)$*.*

Proof. Suppose not. Then by compactness and 2.39 we can find b_j s.t. $\models \phi(a_i, b_j)$ for $i \leq j$ and $\models \neg\phi(a_i, b_j)$ for $i \geq j + 1$, so $\phi(x, y)$ is unstable, a contradiction. □

Lemma 2.44. *Let* $p \in S_x(A)$ *be stationary,* $M \supseteq A$ *sufficiently saturated,* $(a_i : i < \omega$ *a Morley sequence in* p *inside* M*. Let* $p'(x) \in S_x(M)$ *be the unique nonforking extension of* p *over* M*. Then* $p'(x)$ *is definable over* $\{a_i : i < \omega\}$*. (So* $Cb(p) = Cb(p') \subseteq dcl(a_i : i < \omega)$*).*

Proof. Fix $\phi(x, y)$. Let k be from 2.43. Claim: for $b \in M$, $\phi(x, b) \in p$ iff for at least $k + 1$ many i's between between 0 and $2k$ we have $\models \phi(a_i, b)$.

(Note this will give the ϕ-definition of p explicitly as a pointwise Boolean combination of $\phi(a_i, y)$, i in between 0 and $2k$).

Proof of claim: Suppose the right hand side holds. Then by lemma 2.43 this implies that $\models \phi(a_i, b)$ for infinitely many i's. So by 2.41 there is an i such that a_i is independent from b over A and $\models \phi(a_i, b)$, so a_i realizes the unique nonforking extension of p over A, b. But $p'|_{A,b}$ is also the unique nonforking extension of p over A, b so $\phi(x, b) \in p'$.

Other direction: suppose the right hand side fails. Then for at least $k+1$ i's between 0 and $2k$ we have $\models \neg\phi(a_i, b)$, so as above we have $\neg\phi(a_i, b) \in p$. □

2.6 So-called Stable Embeddability

If X is a $\varnothing$-definable set we write $X^{\mathrm{eq}} = \{c \in M^{\mathrm{eq}} : c \in \mathrm{dcl}^{\mathrm{eq}}(X)\}$ (where $c = \bar{a}/E$ for $\bar{a} \in X$ and E $\varnothing$-definable).

Proposition 2.45. *Let X be $\varnothing$-definable and $\bar{b}$ a finite tupe from $\bar{M}$.*

1. $\mathrm{tp}_{\bar{M}}(\bar{b}/X)$ *is definable (i.e for any $\varphi(\bar{x}, \bar{y}) \in L$ there is a $\psi(\bar{y})$ with parameters from X so that $\{\bar{c} \in X : \models \varphi(\bar{b}, \bar{c})\}$ is defined by $\psi(\bar{y})$).*
2. *There is a (finite) $\bar{a} \in \mathrm{dcl}(\bar{b}) \cap X^{\mathrm{eq}}$ such that $\mathrm{tp}(\bar{b}/\bar{a}) \models \mathrm{tp}(\bar{b}/X)$ (i.e $\mathrm{tp}(\bar{b}/\bar{a}) \models \mathrm{tp}(\bar{b}/\bar{c})$ for all $\bar{c} \in X$).*

Proof. We could use the fact that any complete type over any set A is definable (using local 2-rank- see Proposition 2.19 in [10]) but instead we will use what we've done in the course so far.

Choose $\theta(\bar{x}, \bar{c})$ over X ($\bar{c} \in X$) of least Morley rank and degree ($= \alpha, d$) such that $\models \theta(\bar{b}, \bar{c})$ (i.e $\theta(\bar{x}, \bar{c}) \in \mathrm{tp}(\bar{b}/X)$). Notice that for any L-formula $\delta(\bar{x}, \bar{y})$, for any $\bar{d} \in X$ $\delta(\bar{x}, \bar{d}) \in \mathrm{tp}(\bar{b}/X)$ if and only if $(RM, dM)(\theta(\bar{x}, \bar{c}) \wedge \delta(\bar{x}, \bar{d})) = (\alpha, d)$. We denote the right hand side of this statement by $(*)$.

Claim 1. $(*)$ is a definable condition on $\bar{d}$, (i.e there is a $\psi(\bar{y})$ with parameters such that for all $\bar{d} \in \bar{M} \models \psi(\bar{d}) \iff (*)$ holds. In which case $\psi(\bar{y})$ is in fact over $\bar{c}$.)

Proof of Claim 1. Let $\theta_1(\bar{x}), \ldots, \theta_d(\bar{x})$ be formulas over maybe additional parameters which partition $\theta(\bar{x}, \bar{c})$ and have $RM = \alpha, dM = 1$. Note that $(*)$ holds if and only if $(RM, dM)(\theta_i(\bar{x}) \wedge \delta(\bar{x}, \bar{d})) = (\alpha, 1)$ for $i = 1, \ldots, d$. But note that for a given $i = 1, \ldots, d$ this is a definable condition on $\bar{d}$ by the definability of the unique $p_i(\bar{x}) \in S_x(\bar{M})$ of Morley rank α containing $\theta_i(\bar{x})$.

This proves part 1. Part 2 is an elaboration: by part 1 for each $\delta(\bar{x}, \bar{y})$ there is a $\psi_\delta(\bar{y})$ over $\bar{c}$ given by the claim. Remember for $\bar{d} \in X$ $\delta(\bar{x}, \bar{d}) \in \mathrm{tp}(\bar{b}/X) \iff \models \psi_\delta(\bar{d})$. Let $A = \{\ulcorner\psi_\delta(\bar{y})\urcorner : \delta\}$, so $A \in \mathrm{dcl}^{\mathrm{eq}}(\bar{c}) \cap X^{\mathrm{eq}}$, and $\mathrm{tp}(\bar{b}/X)$ is definable over A.

Claim 2.

(a) $\text{tp}(\bar{b}/A) \models \text{tp}(\bar{b}/X)$

(b) $A \subseteq \text{dcl}^{\text{eq}}(\bar{b})$.

Proof of Claim 2. Let $\bar{b}'$ realize $\text{tp}(\bar{b}/A)$. Let $\alpha \in \text{Aut}(\bar{M}/A)$ such that $\alpha(\bar{b}) = \bar{b}'$. Then $\alpha(X) = X$ so $\text{tp}(\bar{b}'/X)$ is definable over A by the same defining scheme. So $\text{tp}(\bar{b}'/X) = \text{tp}(\bar{b}/X)$. This proves part (a).

Part (b) is again by automorphism. Now let $\alpha \in \text{Aut}(\bar{M}/\bar{b})$. Again $\alpha(X) = X$ but A is the collection of canonical parameters for $\text{tp}(\bar{b}/X)$ so $\alpha(A) = A$ pointwise. So $A \subseteq \text{dcl}^{\text{eq}}(\bar{b})$.

Claim 3. $A = \text{dcl}^{\text{eq}}(\bar{a})$ for some finite $\bar{a} \subset A$.

Proof of Claim 3. Remember the original $\theta(\bar{x}, \bar{c})$ of minimal Morley rank and degree in $\text{tp}(\bar{b}/X)$. By Claim 2 $\text{tp}(\bar{b}/A) \models \text{tp}(\bar{b}/\bar{c})$. and $A \subseteq \text{dcl}^{\text{eq}}(\bar{c})$. Note that $\text{tp}(\bar{b}/A)$ has rank α and degree d as $\text{tp}(\bar{b}/A) \models \text{tp}(\bar{b}/X)$. Since $\text{tp}(\bar{b}/A) \models \theta(\bar{b}, \bar{c})$ and $A \subseteq X^{\text{eq}}$ we can choose a formula $\chi(\bar{x}, \bar{a}) \in \text{tp}(\bar{b}/A)$ of $(RM, dM) = (\alpha, d)$. Then $A \subseteq \text{dcl}^{\text{eq}}(\bar{a})$ as $\text{tp}(\bar{b}/X)$ is definable over $\bar{a}$ as in Claim 1. □

A similar argument shows that for all small $B \subseteq \bar{M}^{\text{eq}}$ and $\bar{b}$, $\text{dcl}(\bar{b}) \cap B$ is the definable closure of a finite subtuple of $\text{dcl}(\bar{b}) \cap B$.

2.7 Definable groups

Definition 2.46. By a definable group G we mean a definable set equipped with a group operation whose graph is also definable.

We say that G is A-definable or defined over A if all the data is given by formulas over A (note then that the identity is in $\text{dcl}(A)$, inversion is also A-definable).

Remark. 1. Definable bijections preserve Morley rank and degree.

2. Let $\bar{b} \in \text{acl}^{\text{eq}}(a, A)$. Then $RM(\text{tp}(\bar{b}/A) \leq RM(\text{tp}(\bar{a}/A))$.

The main of example of 1 is: let G be a definable group, $g \in G$. Then for all definable $X \subseteq G$, $(RM, dM)(X) = (RM, dM)(gX) = (RM, dM)(Xg)$.

Lemma 2.47. *Let G be a definable group. Then we have the descending chain condition on definable subgroups. That is, there is no infinite descending chain $H_1 \gneqq H_2 \gneqq H_3 \gneqq \dots$ of definable subgroups.*

Proof. Using part 1 of the remark, given $H_1 \lneqq H_2$ definable subgroups of G, either $RM(H_1) < RM(H_2)$ (when H_1 has infinite index in H_2), or $RM(H_1) = RM(H_2)$ but $dM(H_1) < dM(H_2)$ (when H_1 has finite index in H_2. □

Note the conclusion: G has a smallest definable subgroup of finite index which we call G^0. Note G^0 is normal and A-definable if G is A-definable.

Still working in complete ω-stable T, with $\bar{M}$ a saturated model of cardinality κ, fix a definable group G over $\varnothing$ with $RM = \alpha, dM = d$. That is, $\alpha = \max\{RM(p(x) : p(x) \in S_G(A)\}$ (where $S_G(A)$ is the set of complete types $p(x)$ over A such that $p(x) \models x \in G$).

Fix any $M \prec \bar{M}$. By Corollary 2.32 there exist distinct $p_1(x), \ldots, p_d(x) \in S_G(M)$ such that $(RM(p_i(x)), dM(p_i(x))) = (\alpha, 1)$, so note that each $p_i(x)$ does not fork over the empty set as $p_i \restriction \varnothing$ has $RM = \alpha$ too.

By Lemma 2.30 each p_i is definable over $\operatorname{acl}^{\text{eq}}(\varnothing)$. Let $p'_i = p_i \restriction \operatorname{acl}^{\text{eq}}(\varnothing)$, which has $RM = \alpha, dM = 1$ by Proposition 2.31.

Let $S = \{p'_1(x), \ldots, p'_d(x)\}$. Note that $\operatorname{acl}^{\text{eq}}(\varnothing) \subseteq \operatorname{dcl}^{\text{eq}}(M) = M^{\text{eq}}$. Let's try to define an action of G on S.

Definition 2.48. Let $g \in G$. Let $p'_i(x) \in S$. Let a realize p'_i and a be independent with g over $\operatorname{acl}^{\text{eq}}(\varnothing)$. Let $b = g \cdot a$.

By the unnumbered remark $RM(\operatorname{tp}(b/g \operatorname{acl}^{\text{eq}}(\varnothing))) = \alpha$ so $RM(\operatorname{tp}(b/\operatorname{acl}^{\text{eq}}(\varnothing))) = \alpha$. So $\operatorname{tp}(b/\operatorname{acl}^{\text{eq}}(\varnothing)) = p'_j$ for some $j \in 1, \ldots, d$. We define $g \cdot p'_i = p'_j$.

Remark 2.49. We could equally as well define an action of $G(M)$ on $\{p_1(x), \ldots, p_d(x)\}$ using $g(\operatorname{tp}(a/M)) = \operatorname{tp}(g \cdot a/M)$.

Exercise 2.50. Show that Definition 2.48 gives a group action of G on S.

Theorem 2.51. *1. The action of G on S is transitive.*

2. For each $i = 1, \ldots, d$ $\operatorname{Stab}(p'_i) := \{g \in G : g \cdot p'_i = p'_i\} = G^0$.

3. The index of G^0 in G is d and each coset of G^0 in G contains a unique p'_i.

Note: for (3), every coset of G^0 in G is $\operatorname{acl}^{\text{eq}}(\varnothing)$ definable, so every $p(x) \in S_G(\operatorname{acl}^{\text{eq}}(\varnothing))$ determines a coset of G^0 in G and (3) says that for $p'_i \in S$, p'_i is determined by its coset.

Proof. 1. Choose $p'_i, p'_j \in S$. Let a_i realize p'_i and a_j realize p'_j such that a_i is independent with a_j over $\operatorname{acl}^{\text{eq}}(\varnothing)$. Let $g = a_j a_i^{-1} \in G$, so $g \cdot a_i = a_j$. Then g is independent with each of a_i and a_j over $\operatorname{acl}^{\text{eq}}(\varnothing)$.

(Why? $RM(\operatorname{tp}(a_j/a_i \operatorname{acl}^{eq}(\varnothing))) = \alpha$ and $RM(\operatorname{tp}(a_j/a_i^{-1} \operatorname{acl}^{\text{eq}}(\varnothing))) = \alpha$ so by the unnumbered remark $RM(\operatorname{tp}(g/a_i^{-1} \operatorname{acl}^{\text{eq}}(\varnothing))) = \alpha$ so $RM(\operatorname{tp}(g/\varnothing) = \alpha$. So g is independent with a_i^{-1} over the empty set and therefore independent with a_i over the empty set. By symmetry a_i is independent with g over the empty set. Likewise for a_j.)

Then by definition of the group action, $g \cdot p'_i = p'_j$.

2. Let $X_1 = G^0, X_2, \dots, X_s$ be cosets of G^0 in G (defined over $\mathrm{acl}^{\mathrm{eq}}(\varnothing)$). Note that for each $i = 1, \dots, s$ $G^0 = \{g \in G : gX_i = X_i\}$. Also note that since G^0 is normal in G, the X_i's are both left and right cosets of G^0.

   Claim. $\mathrm{Stab}(p'_i(x))$ is definable over $\mathrm{acl}^{\mathrm{eq}}(\varnothing)$.

   Proof of Claim. (We may assume M is big). Choose $g \in G(M)$ and notice that $g \cdot p_i = p'_i$ if and only if $gp_i = p_i$ ($gp_i = \mathrm{tp}(ga/M)$ where $a \models p_i$).

   Choose a formula $\varphi(x) \in p_i(x)$ of $RM = \alpha$ and $dM = 1$. Note that for $g \in G(M)$, $gp_i = p_i$ if and only if $g \cdot \varphi \in p_i$ (as there exists a unique complete type over M of $RM = \alpha$ and $dM = 1$ containing φ). By definability of p_i, the set of such $g \in G(M)$ is definable over $\mathrm{acl}^{\mathrm{eq}}(\varnothing)$. This proves the claim.

   As G act transitively on S, $\mathrm{Stab}(p'_i)$ has finite index in G and thus contains G^0. However $g \in \mathrm{Stab}(p'_i)$ implies $gX_i = X_i$, so $g \in G^0$ and $\mathrm{Stab}(p'_i) = G^0$.

3. Since $\mathrm{Stab}(p'_i) = G^0$ for some/any $i = 1, \dots, d$, we have $|S| = |G/G^0|$ by the orbit-stabilizer theorem $(*)$.

   Each of $p_1(x), \dots, p_d(x) \in S_G(M)$ determine some X_i as $g \in G$ implies $g \in X_1 \vee \cdots \vee g \in X_s$. On the other hand for any $p_i(x) \in S_G(M)$ and X_j there is a $g \in G$ with $gp_i \in X$ and $gp_i = p_j$ for some j as it has the same RM, dM. So $s \leq d$, and by $(*)$ $s = d$. So each coset of G^0 in G is contained in a unique p_i or p'_i.

□

Corollary 2.52. *G is connected ($G = G^0$) if and only if $dM(G) = 1$.*

Lemma 2.53. *Let $X \subseteq G$ be definable with parameters. Then $RM(X) = \alpha$ if and only if finitely many left/right translates of X cover G.*

Proof. ($\Leftarrow$) Each translate of X has the same $RM = \beta$ as X. The right hand side implies $X \cup g_1 X \cup \dots \cup g_k(X) = G$, so $\beta = \alpha$.

($\Rightarrow$) Fix a model M with X over M. Let $p_o(x) \in S_G(M)$ be the unique type of $RM = \alpha$ such that $p_0(x) \vdash x \in G^0$. Note that $(x \in X) \in p(x)$ for $p(x) \in S_G(M)$ of $RM = \alpha$. But by an earlier proposition, there is a $g \in G(M)$ such that $gp(x) = p_0(x)$, so $\exists g \in gM$ such that $gX \in p_0$, $p_0(x) \vdash (x \in gX)$. So we may assume $p_0(x) \vdash (x \in X)$.

It is enough to show finitely many (left) translates of X cover G^0. Let $g \in G^0$. Let $(a_i : i < \omega)$ be a Morley sequence in $p_0(x)$, so $\{a_1, a_2, \dots\}$ is independent over M. By Lemma 2.41 and symmetry a_i is independent with g over M for some $i < \omega$, so a_i^{-1} is also independent with g over M. By routine computation (left as an exercise) $RM(\mathrm{tp}(a_i^{-1}g)/M) = \alpha$ as $a_i^{-1}g \in G^0$. $a_i^{-1}g$ realizes p_0 so $a_i^{-1}g \in X$ and $g \in a_i X$.

So we have proved that any $g \in G$ is in $a_i X$ for some $i = 1, 2, \ldots$. By compactness there exist $a_1, \ldots, a_n$ such that any $g \in G^0$ is in $a_1 X \cup \cdots \cup a_n X$, so the right hand side covers G_0. The same argument works for right translates. $\square$

Definition 2.54. No assumptions on T except that it is complete and $\bar{M} \models T$. Let G be a definable group and $X \subseteq G$ definable. We call X left/right generic if finitely many left/right translates of X cover G. So for T ω-stable, left generic and right generic are equivalent to each other and to having maximal Morley rank.

Chapter 3

Definable automorphism groups (aka binding groups)

We assume T is ω-stable.

Definition 3.1. Let G be a group. A left torsor (left principal homogeneous space) for G is a set X together with a left action $\alpha : G \times X \to X$ of G on X which is regular i.e strictly transitive i.e $\forall x, y \in X\ \exists^{=1} g \in G\ (gx = y)$. The action α is $\alpha(g, x) = gx$. This is also known as being transitive and free.

Definition 3.2. By a bitorsor we mean a tuple (G, S, H), left action of G on S, and right action of G on S such that S is a left G torsor and a right H torsor, and $(gx)h = g(xh)$ for all $g \in G, x \in S, h \in H$.

Remark 3.3.

1. Given such a bitorsor (G, S, H) and $a \in S$ the map f taking $g \in G$ to the unique $h \in H$ such that $ga = ah$ is a group isomorphism (given $g, g' \in G$ we have $(gg')a = af(gg') = af(g)f(g')$).
2. Given an L-structure M and $A \subseteq$, by an A-definable left/right/bi- torsor we mean a left/right/bi- torsor such that all the data are A-definable sets in M.

Note: given e.g a torsor (G, S) definable in M there exists $a \in S(M)$ with the map $g \mapsto ga$ a bijection of G with S defined over M. This induces S with an M-definable group structure isomorphic to G.

Definition 3.4. T ω-stable.

1. Fix a small set A. Fix $\Sigma(x)$ a partial type over A. Let $\mathcal{C}$ be an A-definable set. We say $\Sigma(x)$ is intenal to $\mathcal{C}$ (or $\mathcal{C}$ internal) if there exists a small $B \supseteq A$ such that $\Sigma(\bar{M}) \subseteq \mathrm{dcl}(B, \mathcal{C})$, i.e for every $a \models \Sigma$ there is a $\bar{c} \in \mathcal{C}$ such that $a \in \mathrm{dcl}(B\bar{c})$.

2. Given $\Sigma, A, \mathcal{C}$ as in 1. By $\mathrm{Aut}(\Sigma/A, \mathcal{C})$ we mean the group of permutations of $\Sigma(\bar{M})$ induced by $\sigma \in \mathrm{Aut}(M)$ which fix $A, \mathcal{C}$ pointwise.

Remark 3.5. 1. We typically take $A = \varnothing$.

2. When T is ω-stable, $\mathrm{Aut}(\Sigma/A, \mathcal{C})$ coincides with those permutations f of $\Sigma(\bar{M})$ such that for all $\bar{a} \subseteq \Sigma(\bar{M})$ $\mathrm{tp}(\bar{a}/A, \mathcal{C}) = \mathrm{tp}(f(\bar{a}/A, \mathcal{C}$. This was Proposition 2.45 and a back and forth argument, i.e we have to show that any such f lifts to an automorphism of $\bar{M}$ fixing $A, \mathcal{C}$.

3. The binding group theorem says that when Σ is internal to $\mathcal{C}$ there is an A-definable group G and A-definable action on $\Sigma(\bar{M})$ isomorphic to the action of $\mathrm{Aut}(\Sigma/A, \mathcal{C})$ on $\Sigma(\bar{M})$.

4. In a somewhat special case which we can reduce to, there is a canonical A-definable bitorsor (G, S, H) attached to the internality data $H \subseteq \mathcal{C}^{\mathrm{eq}}$ defined over A.

Example 3.6. 1. Consider the 2-sorted (saturated) structure $M = (G, X)$ where $(G, \cdot)$ is a group and $\alpha : X \times G \to X$ is a regular right action on X. X is internal to G (we need a parameter $b \in X$ to see this; over b, X is in bijection with G), but $\mathrm{Aut}(M/G)$ acts transitively on X. Letting $\mathcal{G} = \mathrm{Aut}(X/G)$, $(\mathcal{G}, X, G)$ is a bitorsor.

2. Consider the 2-sorted (saturated) structure $M = (F, V)$ where F is an algebraically closed field (with the ring structure), V is an n-dimensional F-vector space, and $\alpha : F \times V \to V$ is the action representing scalar multiplication. $T = \mathrm{Th}(M)$ is ω-stable. V is internal to F because if $\bar{v} = (v_1, \ldots, v_n)$ is a basis of V/F then $V \subseteq \mathrm{dcl}(\bar{v}, F)$. Think of F acting on the right of V, we make a few observations:

   (a) Given bases $\bar{v} = (v_1, \ldots, v_n)$ and $\bar{w} = (w_1, \ldots, w_n)$ there is an $n \times n$ (nonsingular) matrix over F taking $\bar{v}$ to $\bar{w}$.

   (b) $\mathrm{Aut}(M/F)$ acts regularly on the set Q of bases (Q is $\varnothing$-definable and isolates a complete type over $\varnothing$). We can check that given $\bar{v}, \bar{w} \in Q$ there is a unique $f \in \mathrm{Aut}(M)$ such that $f(\bar{v}) = \bar{w}$.

   (c) What is $\mathrm{Aut}(V/F)$? Given $f \in \mathrm{Aut}(V/F)$ we know f fixes each element of F, f preserves $+$, f is invertible, and for $r \in F, v \in V$ we have $f(rv) = rf(v)$. So f is an invertible linear transformation. Thus $\mathrm{Aut}(V/F) = GL(V)$.

   The bitorsor is $(GL(V), Q, GL_n(F)$.

3. We state the following theorem without proof. Let T be countable and uncountably categorical, so ω-stable. $\bar{M}$ is a big model. Obviously there exist strongly minimal definable sets X (i.e $(RM, dM)(X) = (1, 1)$).

   Fact. We can find such X defined over the prime model $M_0 \models T$ (T is 1-sorted).

Let's assume there exist such $\mathcal{C}$ defined over $\varnothing$.

Fact. For all $a \in \bar{M}$ there exist $a_1, \ldots, a_n \in \mathrm{dcl}^{\mathrm{eq}}(a)$ such that $a \in \mathrm{acl}^{\mathrm{eq}}(a_1, \ldots, a_n)$ and for all $i = 1, \ldots, n$ $\mathrm{tp}(a_{i+1}/a_1 \ldots a_n)$ is $\mathcal{C}$-internal.

T is called almost strongly minimal if for some strongly minimal definable set $\mathcal{C}$ (even with parameters) there is a finite A such that $\bar{M} \subseteq \mathrm{acl}(A, \mathcal{C})$. So we can deduce from the binding group theorem with a bit more that if T is not almost strongly minimal then there exists an infinite definable group.

A basic example of a non- almost strongly minimal $\aleph_1$-categorical T is $\mathrm{Th}((\mathbb{Z}/4\mathbb{Z})^\omega, +)$. For another example, take an algebraically closed field $(F, +, \cdot)$ and the projection $X \twoheadrightarrow F$ such that each fiber $\pi^{-1}(a)$ is a left torsor for $(F, +)$ $\varnothing$-definably and uniformly. This is also $\aleph_1$-categorical and not almost strongly minimal.

As before, T is ω-stable and complete, and $\bar{M} \models T$ is a big model.

We assume $\Sigma(\bar{x})$ is a partial type over $\varnothing$, $\mathcal{C}$ is a $\varnothing$-definable set, and $\Sigma(\bar{x})$ is internal to $\mathcal{C}$. Our long term goal is to construct a $\varnothing$-definable group G and a $\varnothing$-definable action of this G on $\Sigma(\bar{M})$, isomorphic to the action of $\mathrm{Aut}(\Sigma/\mathcal{C})$ on $\Sigma(\bar{M})$ (note that $\Sigma(\bar{M})$ is type definable, so what we mean by this is the following: there is some (partial) $\varnothing$-definable function $\alpha(\cdot,\cdot)$ such that for all $g \in G$ and $a \in \Sigma(\bar{M})$, $\alpha(g, a)$ is defined and it realizes $\Sigma(\bar{x})$).

Before proceeding to the construction of such a $\varnothing$-definable group G and a $\varnothing$-definable action of this G on $\Sigma(\bar{M})$, we first need to do some preparation and reductions.

By definition of internality, there is a small B such that $\Sigma(\bar{M}) \subseteq \mathrm{dcl}(B, \mathcal{C})$, i.e., for all $a \in \Sigma(\bar{M})$, there is some $\bar{b} \in B$ and $\bar{c} \in \mathcal{C}$ such that $a \in \mathrm{dcl}(\bar{b}, \bar{c})$.

Lemma 3.7. *This B can be chosen to be a finite tuple, say $\bar{b}$.*

Proof. Note that

$$\Sigma(\bar{x}) \models \bigvee_{\substack{f(\cdot,\cdot)\ \varnothing\text{-definable}, \\ \bar{b} \subseteq B}} (\exists \bar{c} \in \mathcal{C})(\bar{x} = f(\bar{b}, \bar{c})). \tag{3.1}$$

Therefore, by compactness, we can replace the right hand side of (3.1) by a finite subdisjunction. □

Lemma 3.8. *We may even choose this $\bar{b}$ to be a (finite) tuple of realizations of $\Sigma(\bar{x})$ (often called a "fundamental system" of solutions).*

Proof. By a slight generalization of Proposition 2.45, there are finite tuples $\bar{b}_0 \subseteq \Sigma(\bar{M}), \bar{c}_0 \subseteq \mathcal{C}$ such that $\mathrm{tp}(\bar{b}/\bar{b}_0, \bar{c}_0) \models \mathrm{tp}(\bar{b}/\Sigma(\bar{M}) \cup \mathcal{C})$.

Claim. It follows that any $a \in \Sigma(\bar{M})$ is in $\mathrm{dcl}(\bar{b}_0, \mathcal{C})$.

Proof of Claim. Fix $a \in \Sigma(\bar{M})$. It follows that there is $\bar{c} \subseteq \mathcal{C}$ such that $a = f(\bar{b}, \bar{c})$, where $f(\cdot,\cdot)$ is a (partial) $\varnothing$-definable function ($\bar{b}$ is the $\bar{b}$ from Lemma 3.7). Let $\alpha \in \mathrm{Aut}(\bar{M}/\bar{b}_0, \bar{c}_0, \bar{c})$.

We want to show that $\alpha(a) = a$. Note that $\alpha(\bar{b})$ has the same type over $\Sigma(\bar{M}) \cup \mathcal{C}$ as $\bar{b}$. Since $f(\bar{b}, \bar{c}) = a$, on the one hand we get $f(\alpha(\bar{b}), \bar{c}) = a$. On the other hand, $\alpha(a) = f(\alpha(\bar{b}), \bar{c})$. Hence, $\alpha(a) = a$, and $a \in \mathrm{dcl}(\bar{b}_0, \bar{c}_0, \bar{c})$ follows. □

Consequently, we may find some finite tuple $\bar{b}$ as in Lemma 3.7 that also satisfies $b \subseteq \Sigma(\bar{M})$.

Lemma 3.9. *There is $\varphi(\bar{x}) \in \Sigma(\bar{x})$ which is internal to $\mathcal{C}$, and $\mathrm{Aut}(\varphi/\mathcal{C})$ is canonically isomorphic to $\mathrm{Aut}(\Sigma, \mathcal{C})$, and even more.*

Proof. By compactness, there is some partial $\varnothing$-definable function $f(\cdot, \cdot)$ such that $\Sigma(\bar{x}) \models (\exists \bar{c} \subseteq \mathcal{C})(f(\bar{b}, \bar{c}) = x)$. Again, by compactness, there is some $\varphi(\bar{x}) \in \Sigma(\bar{x})$ such that $\varphi(\bar{x}) \models (\exists \bar{c} \subseteq \mathcal{C})(f(\bar{b}, \bar{c}) = x)$. Therefore, this φ is internal to $\mathcal{C}$. Note that if $\sigma \in \mathrm{Aut}(\varphi/\mathcal{C})$, then we have $\sigma \restriction \Sigma \in \mathrm{Aut}(\Sigma, \mathcal{C})$. However, σ is in fact determined by its restriction to Σ. Pourquoi? Assume $a \models \varphi$ and find some $\bar{c} \in \mathcal{C}$ with $a = f(\bar{b}, \bar{c})$. Then $\sigma(\bar{a}) = f(\sigma(\bar{b}), \bar{c})$. This means that σ is determined by $\sigma(\bar{b})$, which is in turn determined by $\sigma \restriction \Sigma$.

We get even more: The action of $\mathrm{Aut}(\varphi/\mathcal{C})$ on Σ is by restriction, so, if its action on φ were $\varnothing$-definable, so would be its action on Σ. We conclude that one can replace $\Sigma(\bar{x})$ by $\varphi(\bar{x})$, i.e., if one can prove that $\mathrm{Aut}(\varphi/\mathcal{C})$ (and its action) is $\varnothing$-definable, then one can prove the same for $\mathrm{Aut}(\Sigma, \mathcal{C})$ as well. □

Thus, from now on, it is justified to restrict our attention to $\mathrm{Aut}(\varphi, \mathcal{C})$; and we shall do so.

Lemma 3.10. *We may rechoose the finite tuple $\bar{b}$ of Lemma 3.8 to be a tuple of realizations of $\varphi(\bar{x})$ inside the prime model M_0 of T (note that the $\bar{b}$ from Lemma 3.8 is still good for $\varphi(\bar{x})$ by the proof of Lemma 3.9; recall that we had $\varphi(\bar{x}) \models (\exists \bar{c} \subseteq \mathcal{C})(f(\bar{b}, \bar{c}) = x)$).*

Proof. Define $r(\bar{y}) = \mathrm{tp}(\bar{b}/\varnothing)$. In particular, $r(\bar{y}) \models (\forall x)(\varphi(\bar{x}) \to (\exists \bar{c} \subseteq \mathcal{C})(f(\bar{y}, \bar{c}) = \bar{x}))$. So we can choose $\psi(\bar{y}) \in r(\bar{y})$ such that $\psi(\bar{y}) \models (\forall \bar{x})(\varphi(\bar{x}) \to (\exists \bar{c} \subseteq \mathcal{C})(f(\bar{y}, \bar{c}) = \bar{x}))$. By possibly strengthening $\psi(\bar{y})$, we may assume that $\psi(\bar{y}) \models \bigwedge_{1 \leq i \leq |\bar{y}|} \varphi(y_i)$. Let the new $\bar{b}$ realize $\psi(\bar{y})$ in M_0. □

Now let $q = \mathrm{tp}(\bar{b}/\varnothing)$, which is an isolated type, and let Q denote the set of realizations of q.

Remark 3.11. (a) q is also internal to $\mathcal{C}$.

(b) Moreover, for any $\bar{b}' \models q$, we have $\bar{b}' \in \mathrm{dcl}(\bar{b}, \mathcal{C})$.

(c) In fact, for any $\bar{b}', \bar{b}'' \models q$, we have $\bar{b}'' \in \mathrm{dcl}(\bar{b}', \mathcal{C})$, as $\mathrm{tp}(\bar{b''}) = \mathrm{tp}(\bar{b}')$ and q is isolated.

Proof. The first part is straightforward. For the second part, take any $\bar{b}' \models q$. Each coordinate of $\bar{b}'$ is in $\varphi(\bar{M})$, and so in $\mathrm{dcl}(\bar{b}, \mathcal{C})$ as well. Therefore, so is $\bar{b}'$. □

Lemma 3.12. $\mathrm{Aut}(q/\mathcal{C})$ *and* $\mathrm{Aut}(\varphi/\mathcal{C})$ *are canonically isomorphic, and the $\varnothing$-definability of the groups and the relevant actions is equivalent.*

Proof. Any $\sigma \in \mathrm{Aut}(\varphi/\mathcal{C})$ restricts to some $\sigma' = \sigma \restriction Q \in \mathrm{Aut}(q/\mathcal{C})$. But also, σ is determined by σ': Given $a \models \varphi(\bar{x})$, if $\bar{a} = f(\bar{b}, \bar{c})$, then $\sigma(\bar{a}) = f(\sigma(\bar{b}), \bar{c})$.

What about $\varnothing$-definability? Set $G = \mathrm{Aut}(q/\mathcal{C})$. If $\mathrm{Aut}(\varphi/\mathcal{C})$ is $\varnothing$-definable, then the action on q (i.e., on Q) by restriction is also $\varnothing$-definable. Let the action be $\alpha : G \times \varphi(\bar{M}) \to \varphi(\bar{M})$. Then $\langle \alpha, \alpha, \ldots, \alpha \rangle$ yields a $\varnothing$-definable action $G \times Q \to Q$. Conversely, suppose G is $\varnothing$-definable. Given $g \in G$ and $\bar{a} \models \varphi$, choose any $\bar{b}' \models q$ and $\bar{c} \in \mathcal{C}$ such that $\bar{a} = f(\bar{b}', \bar{c})$. Then $g{\cdot}a = f(g{\cdot}\bar{b}', \bar{c})$, and consequently $\mathrm{Aut}(\varphi/\mathcal{C})$ is $\varnothing$-definable. □

This concludes our preparation and the reductions we have needed before we started our construction. In summary, we have reduced the problem to the case where $\Sigma(\bar{x})$ is now a complete isolated type $q(\bar{y})$ over $\varnothing$, and (by rechoosing the partial $\varnothing$-definable function $f(\cdot,\cdot)$), for any two realizations $\bar{b}', \bar{b}''$ of q, we have $f(\bar{b}', \bar{d}) = \bar{b}''$ $(*)$, for some $\bar{d} \subseteq \mathcal{C}$ (by compactness).

Let $\mathcal{G} = \mathrm{Aut}(q/\mathcal{C})$. By Proposition 2.45, we can find $\bar{c} \in \mathcal{C}^{\mathrm{eq}}$ such that $\bar{c} \subseteq \mathrm{dcl}(\bar{b})$ and $\mathrm{tp}(\bar{b}, \bar{c}) \models \mathrm{tp}(\bar{b}, \mathcal{C}^{\mathrm{eq}})$. Define $q_{\bar{c}} = \mathrm{tp}(\bar{b}, \bar{c})$, and let $Q_{\bar{c}}$ be the set of its realizations. Observe that $q_{\bar{c}}$ is also isolated. Note that $\mathcal{G}$ also acts on $Q_{\bar{c}}$ ($\mathcal{G}$ fixes $\mathcal{C}$ pointwise, so it takes a realization of $q_{\bar{c}}$ to another realization of $q_{\bar{c}}$).

Lemma 3.13. *$\mathcal{G}$ acts regularly on $Q_{\bar{c}}$ (i.e., $Q_{\bar{c}}$ is a left torsor for $\mathcal{G}$).*

Proof. Note that $q_{\bar{c}}$ implies a complete type over $\mathcal{C}$. As all realizations of $q_{\bar{c}}$ have the same type over $\mathcal{C}$, $\mathcal{G}$ acts transitively on $Q_{\bar{c}}$. But also, by $(*)$, $\sigma \in \mathcal{G}$ is determined by $\sigma(\bar{b}')$, where $\bar{b}$ is some/any realization of $q_{\bar{c}}$: Fix $\bar{b}'$ and let $\bar{b}'' \in Q_{\bar{c}}$, say $\bar{b}'' = f(\bar{b}', \bar{c})$. But then $\sigma(\bar{b}'') = f(\sigma(\bar{b}'), \bar{c})$. □

Recap. We have reduced our problem to the case of a complete and isolated type over $\varnothing$ $q(\bar{y})$ (q was given by $q(\bar{y}) = \mathrm{tp}(\bar{b}/\varnothing)$), Q is the set of its realizations, $f(\cdot,\cdot)$ is a partial $\varnothing$-definable function, and for all $\bar{b}', \bar{b}'' \in Q$, there is some $\bar{d} \subseteq \mathcal{C}$ such that $f(\bar{b}', \bar{d}) = \bar{b}''$. Furthermore, there is some $c \in \mathcal{C}^{\mathrm{eq}}$ with $c \in \mathrm{dcl}(\bar{b})$ and $\mathrm{tp}(\bar{b}/c) \models \mathrm{tp}(\bar{b}/\mathcal{C})$. Moreover, $q_c = \mathrm{tp}(\bar{b}/c)$ is an isolated type, and Q_c is the set of its realizations. In particular, q_c implies a complete type over $\mathcal{C}$. We defined $\mathcal{G} = \mathrm{Aut}(q/\mathcal{C})$, which acts regularly on Q_c (Lemma 3.13).

Proposition 3.14. *There is a c-definable group $H_c \subseteq \mathcal{C}^{\mathrm{eq}}$, and a c-definable right action of H_c on Q_c which makes $(\mathcal{G}, Q_c, H_c)$ a bitorsor.*

Proof. Consider $Y = \{\bar{d} \in \mathcal{C} : \text{for some/any } \bar{b}' \in Q_c, f(\bar{b}', \bar{d}) \in Q_c\}$, which is a c-definable set. Define an equivalence relation E on Y by declaring $\bar{d}E\bar{d}'$ if and only if for some/any $\bar{b}' \in Q_c$, we have $f(\bar{b}', \bar{d}) = f(\bar{b}', \bar{d}')$, which is c-definable. Let $X = Y/E$. Note that, given $\bar{b}' \in Q_c$, the value of $f(\bar{b}', \bar{d})$ depends only on $\bar{d}/E \in X$. By redefining $f(\cdot,\cdot)$, we can make sure that for all $\bar{b}', \bar{b}'' \in Q_c$, there is a unique $d \in X$ such that $f(\bar{b}', \bar{d}) = \bar{b}''$. For $\bar{b}' \in Q_c$ and $d \in X$, write $\bar{b}' \cdot d$ for $f(\bar{b}, \bar{d})$.

Fix $\bar{b}$. Note that for any $d_1, d_2 \in X$, there is a unique $d_3 \in X$ such that $(\bar{b} \cdot d_1) \cdot d_2 = \bar{b} \cdot d_3$ (as $f(\bar{b}, d_1) \cdot d_2 \in Q_c$, there must be a unique d_3 with $f(\bar{b}, d_1) \cdot d_2 = f(\bar{b}, d_3)$). Observe that $d_1 \cdot d_2$ does not depend on $\bar{b}$, but only on the fact that $\bar{b} \in Q_c$. We define $d_1 \cdot d_2$ to be this d_3, and it follows that the operation $\cdot$ is c-definable.

For $\sigma \in \mathcal{G}$, let d_σ be the unique element of X such that $\sigma(\bar{b}) = \bar{b} \cdot d_\sigma (= f(\bar{b}, d_\sigma))$ (so, d_σ depends on σ and b).

We now do the only computation in the proof:

Claim. For $\sigma, \tau \in \mathcal{G}$, we have $d_{\sigma\tau} = d_\sigma \cdot d_\tau$.

Proof of Claim. We have

$$\bar{b} \cdot d_{\sigma\tau} = \sigma\tau(\bar{b}) = \sigma(\tau(\bar{b})) = \sigma(b \cdot d_\tau) = \sigma(\bar{b}) \cdot \sigma(d_\tau) = \sigma(\bar{b}) \cdot d_\tau = (\bar{b} \cdot d_\sigma) \cdot d_\tau, \tag{3.2}$$

as σ fixes $\mathcal{C}$. Thus, $d_{\sigma\tau} = d_\sigma \cdot d_\tau$ as wanted.

Note that the map that sends $\sigma \mapsto d_\sigma$ (depending on $\bar{b}$) is a bijection between $\mathcal{G}$ and X, so it transfers the group structure of $\mathcal{G}$ to a group structure on X, which, by the claim above, precisely $\cdot$. Define $H_c = (X, \cdot)$, which is a c-definable group.

One can check that the map that sends $(\bar{b}', d)$ to $\bar{b}' \cdot d (= f(\bar{b}', \bar{d}))$ gives a right action of H_c on Q_c, making Q_c a right H_c-torsor. As $\mathcal{G}$ fixes $\mathcal{C}$, and so X pointwise, we see that $(\mathcal{G}, Q_c, H_c)$ is a bitorsor: this is because $\sigma(\bar{b} \cdot \bar{d}) = \sigma(\bar{b}) \cdot \sigma(\bar{d}) = \sigma(\bar{b}) \cdot \bar{d}$. □

But we want $\mathcal{G}$ and its action on Q to be $\varnothing$-definable. First,

Lemma 3.15. *$\mathcal{G}$ and its action on Q_c are c-definable.*

Proof. Consider $Q_c^2 = \{(\bar{b}_1, \bar{b}_2) : \bar{b}_1, \bar{b}_2 \in Q_c\}$. For each $(\bar{b}_1, \bar{b}_2) \in Q_c^2$, there is a unique $\sigma \in \mathcal{G}$ such that $\sigma(\bar{b}_1) = \bar{b}_2$. When do two such pairs give the same automorphism σ?

Claim 1. There is a (unique) σ such that $\sigma(\bar{b}_1) = \bar{b}_2$ and $\sigma(\bar{b}_1') = \bar{b}_2'$ if and only if $\mathrm{tp}(\bar{b}_1, \bar{b}_1'/\mathcal{C}) = \mathrm{tp}(\bar{b}_2, \bar{b}_2'/\mathcal{C})$ if and only if there is a (unique) $h \in H_c$ such that $\bar{b}_1' = \bar{b}_1 \cdot h$ and $\bar{b}_2' = \bar{b}_2 \cdot h$.

Proof of Claim 1. This follows from the fact that all elements of Q_c have the same type over $\mathcal{C}$.

So, we have a c-definable equivalence relation E_c on Q_c^2 (given by declaring $(\bar{b}_1, \bar{b}_2) E_c (\bar{b}_1', \bar{b}_2')$ if and only if the unique $\sigma \in \mathcal{G}$ taking $\bar{b}_1$ to $\bar{b}_1'$ takes $\bar{b}_2$ to $\bar{b}_2'$). It follows that Q_c^2/E_c is in bijection with $\mathcal{G}$.

Claim 2. The induced action of Q_c^2/E_c on Q_c is c-definable.

Proof of Claim 2. This directly follows from the observation that, given $(\bar{b}_1, \bar{b}_2)/E_c$ and $\bar{b}_1' \in Q$, there is a unique $\bar{b}_2' \in Q_c$ with $(\bar{b}_1', \bar{b}_2') \in (\bar{b}_1, \bar{b}_2)/E_c$. □

It follows that the group structure on Q_c^2/E_c is also c-definable. Let us call this group G_c. Note that (G_c, Q_c, H_c) is a c-definable bitorsor.

Theorem 3.16. *$\mathcal{G}$ and its action on Q are $\varnothing$-definable.*

Proof. Let us return to the previous partial $\varnothing$-definable function $f(\cdot,\cdot)$, i.e., for all $\bar{b}_1, \bar{b}_2 \in Q$, there is $\bar{d} \subseteq \mathcal{C}$ such that $f(\bar{b}_1, \bar{d}) = \bar{b}_2$. Remember that our c comes from $\mathrm{dcl}^{\mathrm{eq}}(\bar{b})$, so we can write $c = g(\bar{b})$. Let $s = \mathrm{tp}(c/\varnothing)$, and define S to be the set of its realizations. Note that $g : Q \twoheadrightarrow S$, and any $c' \in S$ could have played the role of c (in Lemma 3.15), giving $Q_{c'}, H_{c'}$ etc. Let $Z = \{(\bar{b}_1, \bar{b}_2) \in Q^2 : g(\bar{b}_1) = g(\bar{b}_2)\}$ (i.e., $\bar{b}_1, \bar{b}_2 \in Q_{c'}$ for some unique c'). Again, each $(\bar{b}_1, \bar{b}_2) \in Z$ gives a unique $\sigma \in G$ with $\sigma(\bar{b}_1) = \bar{b}_2$. When do two pairs in Z give the same σ?

Claim. For $(\bar{b}_1, \bar{b}_2), (\bar{b}_1', \bar{b}_2') \in Z$, there is a (unique) σ such that $\sigma(\bar{b}_1) = \bar{b}_2$ and $\sigma(\bar{b}_1') = \bar{b}_2'$ if and only if $\mathrm{tp}(\bar{b}_1, \bar{b}_1'/\mathcal{C}) = \mathrm{tp}(\bar{b}_2, \bar{b}_2'/\mathcal{C})$ if and only if there is $\bar{d} \subseteq \mathcal{C}$ with $f(\bar{b}_1, \bar{d}) = \bar{b}_1'$ and $f(\bar{b}_2, \bar{d}) = \bar{b}_2'$

Proof of Claim. This follows from the fact that $\bar{b}_1, \bar{b}_2$ have the same type over $\mathcal{C}$, as both realize $q_{c'}$ for a unique c', which implies a complete type over $\mathcal{C}$.

So, we obtain a $\varnothing$-definable equivalence relation E on Z such that Z/E is in a bijective correspondence with $\mathcal{G}$. As before, the action of Z/E on Q is $\varnothing$-definable, as is the the group structure on Z/E. This finishes the proof. □

Remark 3.17. From Proposition 3.14 and Lemma 3.15, we see that if $q(\bar{y}) \in S(\varnothing)$ is isolated, $\mathcal{C}$-internal, fundamental (i.e., for all $\bar{b}_1, \bar{b}_2 \models q$, $\bar{b}_2 \in \mathrm{dcl}(\bar{b}_1, \mathcal{C})$), weakly orthogonal to $\mathcal{C}$ (i.e., $q(\bar{y})$ implies a complete type over $\mathcal{C}$), for some/any $\bar{b} \models q$, $\mathrm{dcl}(\bar{b}) \cap \mathcal{C}^{\mathrm{eq}} = \mathrm{dcl}^{\mathrm{eq}}(\varnothing)$, and $\mathcal{G} = \mathrm{Aut}(q/\mathcal{C})$, then we get a $\varnothing$-definable bitorsor (G, Q, H), where Q is the set of realizations of q. such that the action of G on Q is isomorphic to the action of $\mathcal{G}$ on Q, and $H \subseteq \mathcal{C}^{\mathrm{eq}}$. Moreover, the converse is also true: Let (G, Q, H) be a $\varnothing$-definable bitorsor with $H \subseteq \mathcal{C}^{\mathrm{eq}}$, and assume that the action of G preserves types over $\mathcal{C}$. Then Q is equal to the set of realizations of an isolated $\mathcal{C}$-internal type q over $\varnothing$, which is moreover fundamental and weakly orthogonal to $\mathcal{C}$.

For example, we will see the following case later: Let K be a differential field with characteristic 0 with C_K algebraically closed, and assume that $\partial Y = AY$ $(*)$ is a linear differential equation over K, where A is an $(n \times n)$ matrix over K and $Y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$. Let Z be an $(n \times n)$ matrix whose columns form a basis for the C-vector space of solutions to $(*)$ such that $\mathrm{tp}(Z/K)$ is isolated. Then it is fundamental and weakly orthogonal to C (the constants). So we get the K-definable bitorsor (G, Q, H), where H is an algebraic group in the constants.

Chapter 4

$\mathbf{DCF}_0$

4.1 Review of ACF

ACF is the theory of algebraically closed fields in the language L_r of rings $\{+, \times, -, 0, 1\}$. You should have seen in the basic course that ACF has QE in L_r and the completions of ACF are given by fixing the characteristic as 0 or as a prime p.

We will focus a lot on ACF_0 as DCF_0 is an expansion of it in $L_{r,\partial} = \{+, \times, -, 0, 1, \partial\}$. By the way, QE for the incomplete theory ACF (i.e QE across the characteristic) is behind many basic applications, e.g Ax's theorem (a one-to-one polynomial map from $\mathbb{C}^n$ to $\mathbb{C}^n$ is onto).

Lemma 4.1. *ACF_0 is strongly minimal.*

Proof. By QE, any definable $X \subseteq \mathbf{K} \models \text{ACF}_0$ is a Boolean combination of solutions to $P(x)$ where $P(x) \in \mathbf{K}[x]$. If $P(x)$ is nonzero, then it has finitely many solutions in $\mathbf{K}$; otherwise, every element of $\mathbf{K}$ is a solution of $P(x)$. □

Corollary 4.2. *ACF_0 is ω-stable.*

Proof. If $\mathbf{K} \models \text{ACF}_0$ is countable, then there are countably many complete 1-type over $\mathbf{K} = \{\text{tp}(a/\mathbf{K}) \mid a \in \mathbf{K}\} \cup \{\text{the unique unrealized type at } \infty\}$. □

Proposition 4.3. *ACF_0 has elimination of imaginaries (as in Def 1.11).*

For the proof, we go through a couple of lemmas and give another proof later.

Lemma 4.4. *Let T be a strongly minimal complete theory (1-sorted, $(RM, dM)(x = x) = (1, 1)$), suppose $acl(\emptyset)$ in the home sort is infinite (i.e $acl(\emptyset) \cap M$ is infinite for any $M \models T$), then T has weak EI.*

Proof. Let $M \models T$ (maybe saturated), and $e \in M^{eq}$. Say $e = (b_1, ..., b_n)/E$ where E is some $\emptyset$-definable equivalence relation on M^n.

Note that any infinite algebraically closed subset of M is the universe of an elementary substructure of M (why? any formula $\varphi(x)$ over A has either finitely many or infinitely many realizations in M. In either case, $\varphi(M) \cap A \neq \emptyset$. Then use Tarski-Vaught test.)

Let $M_0 = acl^{eq}(e) \cap M$, M_0 is infinite and algebraically closed in M and $M_0 \prec M$. We find $c_1, ..., c_n \in M_0$ s.t $e = (c_1, ..., c_n)/E$.

Write e as $f(\bar{b})$, consider $X_1 = \{x_1 \in M \mid \exists x_2 ... \exists x_n f(x_1, ..., x_n) = e\}$. If X_1 is finite, then it is contained in M_0. If X_1 is cofinite, then $X_1 \cap M_0 \neq \emptyset$ as M_0 is infinite.

Let $c_1 \in X_1 \cap M_0$, $X_2 = \{x_2 \in M \mid \exists x_3 ... \exists x_n f(c_1, x_2, ..., x_n) = e\}$. If X_2 is finite, then X_2 is contained in M_0 as $X_2 \subseteq acl^{eq}(e)$; otherwise, X_2 is cofinite and thus has a point in M_0.

Let $c_2 \in X_2 \cap M_0$, continue to get $(c_1, ..., c_n) \in M_0$ as required. Let $\bar{c} = (c_1, ..., c_n)$, so $e = f(\bar{c}) \in dcl^{eq}(\bar{c})$ and $\bar{c} \in acl^{eq}(e)$.

□

Lemma 4.5 (coding finite sets of tuples). *Let* $\mathbf{K} \models ACF_0$, $e = \{\bar{c}_1, ..., \bar{c}_n\}$ *where* $\bar{c}_i$ *are n-tuples from* $\mathbf{K}$*, then there exists a finite tuple* $\bar{d} \in \mathbf{K}$ *and* $\bar{d}, e$ *are interdefinable over* $\emptyset$ *(i.e* $e \in dcl^{eq}(\bar{d})$*,* $\bar{d} \in dcl^{eq}(e)$*).*

Proof. If $n = 1$, take $\bar{d}$ to be the tuple of coefficients of $(X - c_1) \cdots (X - c_n)$. In general, $\bar{c}_i = (c_{i1}, ..., c_{in})$. Let $P(Z, X_1, ..., X_n)$ be the polynomial $(Z + c_{11}X_1 + ... + c_{1n}x_n) \cdots (Z + c_{m1}X_1 + ... + c_{mn}x_n)$, let $\bar{d}$ ne the coefficients of $P(Z, X_1, ..., X_n)$.

We may assume that $\mathbf{K}$ is saturated.

Claim: An automorphism f fixes e iff it fixes $\bar{d}$

Fixing e implies permuting the factors of P, and thus fixing its coefficients. On the other hand, if f fixes $\bar{d}$, it fixes the polynomial. As $\mathbf{K}[Z, X_1, ..., X_n]$ is a UFD, the factors are the unique irreducible factors of P where coefficients of Z are 1. Thus f permutes these factors of P (i.e f permutes $\bar{c}_1, ..., \bar{c}_m$) and so fixes e.

□

Proof of Prop 4.3. Given $e \in \mathbf{K}^{eq}$, $\mathbf{K} \models ACF_0$. By Lem 4.4, let $\bar{c}$ be the finite tuple from $\mathbf{K}$ s.t $e \in dcl(\bar{c})$ and $\bar{c} \in acl(e)$ as ACF_0 is strongly minimal and acl($\emptyset$) is infinite. Let $\bar{c}_1, ..., \bar{c}_m$ be the realizations of tp($\bar{c}/e$). By Lem 4.5, let $\bar{d}$ be a finite tuple from $\mathbf{K}$ interdefinable over $\emptyset$ with $\{\bar{c}_1, ..., \bar{c}_m\}$. For any automorphism f of $\mathbf{K}$ (may assume $\mathbf{K}$ is saturated), f fixes $\bar{d}$ iff f permutes $\bar{c}_1, ..., \bar{c}_m$ iff f fixes e. Thus $\bar{d}$ is interdefinable with e. □

Definition 4.6. Let T be a complete 1-sorted theory, T is a geometric theory if

(1) acl(-) is a pregeometry on any model M of T.

(2) (elimination of the $\exists^\infty$ quantifier) For each $\varphi(x, \bar{y})$, there exists d_φ s.t for any model $M \models T$ and $\bar{b} \subseteq M$, $|\varphi(x, \bar{b})(M)|$ is finite iff it is no bigger than d_φ.

Remark 4.7. 1. acl(-) is a pregeometry if

- $A \subseteq acl(A)$
- acl(-) is transitive
- (exchange) if $b \in acl(A, c) \setminus acl(A)$, then $c \in acl(A, b)$.

2. We call $a_1, ..., a_n \in M$ algebraically independent over $A \subseteq M$ if $a_i \notin acl(Aa_1, ..., a_{i-1}, a_{i+1}, ..., a_n)$ for any i.

3. From acl(-) being a pregeometry, we have that $M \vDash T$ and $A \subseteq M$, any two maximal algebraically independent over A subtuples of $\bar{c}$ have the same cardinality (using exchange, the same proof as one proves that every two bases of a finite-dimensional vector space have the same cardinality). Moreover, if $\bar{c}_0$ is such a subtuple, $\bar{c} \subseteq acl(\bar{c}_0, A)$. The cardinality if called $dim(\bar{c}/A)$.

4. If T is a geometric theory expanding L_r s.t acl(-) = field-theoretic algebraic closure, we call T a geometric theory of fields (i.e in $M \vDash T$, $a \in acl(A)$ iff there is a monic polynomial $P(x)$ over the field generated by A s.t $P(a) = 0$)

Definition 4.8. Let T be a geometric theory, $M \vDash T$ and $X \subseteq M^n$ definable over $A \subseteq M$, $dim_A(X) = \max\{dim(\bar{c}/A) \mid \bar{c} \in X(N) \text{ where } M \prec N\}$.

Lemma 4.9.

1. *Suppose T is geometric theory, $M \vDash T$, $X \subseteq M^n$ is A-definable, then $dim_A(X)$ doesn't depend on A.*

2. *Let $\varphi(\bar{x}, \bar{y}) \in L$, then there is an L-formula $\psi(\bar{y})$ s.t for any $M \vDash T$, $\bar{b} \in M$, dim($\varphi(\bar{x}, \bar{b})(M)) = d$ iff $M \vDash \psi(\bar{b})$. [The case when $\bar{x} = x$ is elimination of $\exists^\infty$].*

3. *For $\bar{c}$ an n-tuple, $dim(\bar{c}/A) = min\{dim(X) \mid X \in \mathrm{tp}(\bar{c}/A)\}$.*

Proof. For 1, it suffices to that for any $A \subseteq B$, $dim_A(X) = dim_B(X)$. By definition, for any $\bar{c} \in X(N)$ where $M \prec N$, $dim(\bar{c}/A) \geq dim(\bar{c}/B)$. Thus $dim_A(X) \geq dim_B(X)$. For the other direction, we prove by induction on n:

If $n = 1$, the case when $dim_A(X) = 0$ is trivially true; if $dim_A(X) = 1$, then X is not an algebraic formula over A, and thus it can be extended into a non-algebraic type p_B over B. Let c realizes p_B (in particular $c \in X$), then $dim(c/B) = 1$ and thus $1 = dim_A(X) \leq dim_B(X)$

If $n = k + 1$, suppose $dim_A(X) = dim(\bar{c}/A) = m$ where $\bar{c} \in X(N)$ for some $M \prec N$. WLOG, assume that $c_1, ..., c_m$ are algebraically independent over A. Furthermore, we can also assume that c_1 is algebraically independent over B (just like the 1-type case considered above). $X(c_1, x_2, ..., x_{k+1}) \subseteq N^k$. Clearly $dim_{A\cup\{c_1\}}(X(c_1, x_2, ..., x_{k+1})) = m - 1$. By IH, we have $dim_{A\cup\{c_1\}}(X(c_1, x_2, ..., x_{k+1})) = dim_{B\cup\{c_1\}}(X(c_1, x_2, ..., x_{k+1}))$. Thus there exist

$d_2, ..., d_{k+1} \in (X(c_1, x_2, ..., x_{k+1})(N')$ for some $M \prec N'$ s.t it has a $m-1$ subtuple $d_{i,1}, ...d_{i,m-1}$ s.t $d_{i,1}, ...d_{i,m-1}$ are algebraic independent over $B \cup \{c_1\}$. As c_1 is algebraically independent over B by assumption, we also have that $c_1, d_{i,1}, ...d_{i,m-1}$ are algebraically independent over B. Thus $m = dim_A(X) \leq dim_B(X)$. This finishes the induction and the whole proof.

For 2, first note that it suffices to prove that for any $\varphi(\bar{x}, \bar{y}) \in L$, there is an L-formula $\psi(\bar{y})$ s.t for any $M \vDash T$, $\bar{b} \in M$, dim $(\varphi(\bar{x}, \bar{b})) \geq d$ iff $M \vDash \psi(\bar{b})$. We do induction on $|\bar{x}|$.

The case when $|\bar{x}| = 1$ is taken care of by elimination of $\exists^\infty$.

If $|\bar{x}| = k+1$, then $dim(\varphi(\bar{x}, \bar{b})) \geq d$ iff $M \vDash \bigvee_{i=1}^{k+1} \exists^\infty x_i(dim(\varphi(x_1, ..., x_{i-1}, x_{i+1}, ..., x_{k+1}; x_i, \bar{b})) \geq d-1)$ iff $M \vDash \bigvee_{i=1}^{k+1} \exists^\infty x_i(\psi_i(x_i, \bar{b}))$ where ψ_i's are given by IH. Applying elimination of $\exists^\infty$ to $\bigvee_{i=1}^{k+1} \exists^\infty x_i(\psi_i(x_i, \bar{b}))$, we get what we want.

For 3, for any $X \in$ tp$(\bar{c}/A)$, $dim(\bar{c}/A) \leq dim(X)$. Thus dim$(\bar{c}/A) \leq$ min$\{dim(X) \mid X \in$ tp$(\bar{c}/A)\}$. Now suppose $dim(\bar{c}/A) = m$, WLOG we can assume that $c_1, ..., c_m$ are algebraically independent over A. Then for each $1 \leq i \leq m-n$, we have $\vDash \varphi_i(c_1, ..., c_m, c_{m+i}) \wedge \exists_{=k_i} x_{m+i}\varphi_i(c_1, ..., c_m, x_{m+i})$ where φ_i's are formulas over A. Let $X := \bigwedge_{i=1}^{m-n} \varphi_i(x_1, ..., x_m, x_{m+i}) \wedge \exists_{=k_i} x_{m+i}\varphi_i(x_1, ..., x_m, x_{m+i})$, then $X \in tp(\bar{c}/A)$ and clearly $dim(X) \leq m$. Thus dim$(\bar{c}/A) =$ min$\{dim(X) \mid X \in$ tp$(\bar{c}/A)\}$.

□

Note: 3 implies that dim$(\bar{c}/A)$ is witnessed by a formula in tp$(\bar{c}/A)$. On the face of it, "dim" is used in 2 grammatical way as dimension of a tuple over a set and as dimension of a definable set. However, we could define for $p \in S_n(A)$, dim(p)=dim$(\bar{c}/A)$ where $\bar{c}$ realizes p. We could define dim$\Sigma(\bar{x})$ ($\Sigma(\bar{x})$ is partial type) which generalize dim (p) and dim(X).

Remark 4.10. If T is strongly minimal, then T is geometric.

Proof. We only have to prove the exchange property and elimination of $\exists^\infty$. Proving exchange directly is a nice exercise. We can also do it using "symmetry" of independence as follows. Let's work in $\overline{M} \vDash T$, $RM(x = x) = 1$ and $dM(x = x) = 1$. Given A, there is a unique nonalgebraic type $p_A(x) \in S_1(A)$ which has Morley rank 1. For $q(x) \in S_1(B)$, $A \subseteq B$ and $p_A(x) \subseteq q(x)$, q is a forking extension of p_A iff $RM(q) = 0$ (i.e q is "algebraic", $q = tp(c/B)$ where $c \in acl(B)$).

Suppose $a \in acl(A, b) \backslash acl(A)$, tp$(a/A) = p_A(x)$ and tp$(a/A, b)$ is a forking extension of $p_A(x)$. By symmetry, tp$(b/A, a)$ is a forking extension of tp(b/A). In particular, tp$(b/A) = p_A(x)$ and $b \in acl(Aa)$. For elimination of $\exists^\infty$, if there is no finite bound on the finite sizes of $|\varphi(x, \bar{b})(M)|$ as $\bar{b}$ varies, then for each n, there is $\bar{b}_n$ s.t $|\varphi(x, \bar{b}_n)| < \omega$ and $\geq n$, so $|\neg\varphi(x, \bar{b}_n)| = \infty$. By compactness, we find $\bar{b} \in \overline{M}$ s.t both $\varphi(x, \bar{b}_n)(\overline{M})$ and $\neg\varphi(x, \bar{b}_n)(\overline{M})$ are infinite, contradicting strong minimality. □

There are plenty nonstrongly minimal geometric theories (of fields): RCF, Th$(\mathbb{Q}_p)$, any completion of the theory of pseudofinite fields, Th$(\mathbb{Z}, +)$.

For those of you who know U-rank, if T is superstable, then T is geometric iff $x = x$ has U-rank 1 (i.e for any complete type $p \in S_1(A)$, either p is algebraic or any forking extension of p is algebraic)

Proposition 4.11. *Let T be strongly minimal*

(1) For each definable $X \subseteq \overline{M}^n$, $dim(X) = RM(X)$

(2) For complete type $p(\bar{x}) \in S_n(A)$, $dim(p) = dim(\bar{c}/A) = RM(p)$ where $\bar{c} \models p$.

Proof. (2) follows from (1). (1) follows from the definition of Morley rank and Lemma 4.12 which is Lemma 2.6 in [11]. □

Lemma 4.12. *Let $X \subseteq \bar{M}^n$ be definable, then $dim(X) \geq r + 1$ iff there are pairwise disjoint definable subsets $(X_i : i < \omega)$ of X s.t $dim(X_i) \geq r$ for all i.*

Lemma 4.13. *(1) ACF_0 is a geometric theory of fields (i.e $a \in acl(A)$ iff $\exists P(x) \in \mathbb{Q}(A)$ monic s.t $P(a) = 0$ where $\mathbb{Q}(A)$ is the field generated by A)*

(2) $a \in dcl(A)$ iff $a \in \mathbb{Q}(A)$ iff $a = \frac{P(\bar{b})}{Q(\bar{b})}$ where $P(\bar{x}), Q(\bar{x}) \in \mathbb{Q}[\bar{x}]$ and $\bar{b} \in A$ (so definable functions in ACF_0 are piecewise rational)

Proof. For (1), the right-to-left direction is clear. For the other direction, suppose $a \in acl(A)$, $A \subseteq \mathbf{K} \models ACF_0$ ($\mathbf{K}$ saturated), $k = \mathbb{Q}(A)$ and $P(a) \neq 0$ for all $P(x) \in k[x]$. Then $tp(a/k) = P_k(x)$ saying that x is transcendental over k. P has infinitely many realization in $\mathbf{K}$, a contradiction.

For (2), the right-to-left direction is clear. For the other direction, $k = \mathbb{Q}(A) \subseteq dcl(A)$. We show that $k = dcl(A)$ (i.e $k = dcl(k)$). Let $a \in dcl(k)$, $I(a/k)$ be the ideal of $k[x]$ consisting of polynomials vanishing on a. $I \neq (0)$ as $a \in acl(A)$. Let $P(x)$ be the minimal polynomial of a over k (i.e monic polynomial in $I(a/k)$ of minimal degree). As $k[x]$ is PID, $I(a/k)$ is generated by $P(x)$. By QE, $tp(a/k)$ is determined by $I(a/k)$. All solutions of $P(x)$ have the same type as a over k, and so $P(x)$ has degree 1, $a \in k$. □

Discussion: by QE in ACF_0, $\mathrm{tp}(\bar{a}/k)$ is determined by $I(\bar{a}/k)$, i.e $\mathrm{tp}(\bar{a}/k)$ is implied by $\{P(\bar{x} = 0) \mid P \in I(\bar{a}/k)\} \cup \{Q(\bar{x}) \neq 0 \mid Q \notin I(\bar{a}/k)\}$. $I(\bar{a}/k)$ is "positive information" about $\bar{a}$. Moreover $I(\bar{a}/k)$ determine $dim(\bar{a}/k)$. This positive information is precisely algebraic geometry.

Let's now work in ACF_0, $\overline{K}$ big saturated model (universal domain ala Andre Weil).

Definition 4.14. An affine algebraic variety (aka a Zariski closed subset of some $\overline{K}^n$ is the solutions in $\overline{K}^n$ of finitely many polynomial equations $P(\bar{x}) = 0$ for $P \in \overline{K}[\bar{x}]$.

Example: $x_2 = ax_1$.

Fact 4.15. For any field k, the polynomial ring $k[x_1, ..., x_n]$ is Noetherian, i.e ACC on ideals or equivalently any ideal is finitely generated. Hence we have DCC on Zariski closed sets (no infinite descending chain of Zariski-closed subsets of $\overline{K}^n$): let $I(X) = \{P(\bar{x}) \in \overline{K}[\bar{x}] \mid P \text{ vanishes on } X\}$, then $I(X_1) \subsetneqq I(X_2) \subsetneqq ....$

Fact 4.16. Using Hilbert's Nullstellensatz, there is a bijection between Zariski closed subsets of $\overline{K}^n$ and radical ideals of $\overline{K}[\bar{x}]$ (i.e an ideal I s.t f^m implies $f \in I$ for any m).

Given Z closed, $V \subseteq \overline{K}^n$, $I(V) = \{P(\bar{x}) \in k[\bar{x}] \mid P \text{ vanishes on } V\}$. Given ideal I, $V(I)$ is the solution set of a finite set of polynomials and so Zariski closed.

Definition 4.17. Let $V \subseteq \overline{K}^n$ be Zariski closed, V is defined over $k \subseteq \overline{K}$ in the sense of algebraic geometry, if $I(V)$ is generated by polynomials in $k[x_1, ..., x_n]$, equivalently $I(V) = I_k(V) \otimes \overline{K}$ where $I_k(V)$ = set of polynomials in $k[\bar{x}]$ vanishing on V.

Fact 4.18. Any affine variety $V \subseteq \overline{K}^n$ has a unique smallest field of definitions. k also has the feature that any automorphism f of $\overline{K}$ fixes V setwise iff it fixes k pointwise.
Proof Sketch: for details, see the proof of Theorem 3.4 in [8]. Let I be $I(V)$, $R = \overline{K}[\bar{x}]/I$ is a vector space over $\overline{K}$. You may choose a collection M of monomials forming mod I a basis for R. Any other monomial can be written as a $\overline{K}$-linear combination of monomials in M mod I. Let k be the field generated by all the coefficients arising this way.

Corollary 4.19. *Given an affine variety $V \subseteq \overline{K}^n$ and $k \prec \overline{K}$, V is defined over k in the model-theoretic sense iff it is defined over k in the field-theoretic sense.*

Proof. The right-to-left direction is obvious. For the other direction, let k_0 be the smallest field of definition of V, if $f \in Aut(\overline{K})$ fixes k pointwise, then f fixes V setwise, and thus f fixes k_0 pointwise, so $k_0 \subseteq dcl(k) = k$. □

Definition 4.20.

1. An Affine Algebraic Variety $V \subseteq \overline{K}^n$ is absolutely irreducible if we cannot write
$$V = V_1 \cup V_2$$
where V_1 and V_2 are proper Zariski closed subsets of V.

2. Suppose moreover that $V \subseteq \overline{K}^n$ is defined over k. We say that V is $k-$irreducible if we cannot find V_1 and V_2 as above for V_1 and V_2 defined ofer k.

Remark 4.21. V (defined over k) is $k-$irreducible if and only if $I_k(V)$ is prime.

Proof. First assume that V is irreducible. Let $ab \in I_k(V)$ be a non-zero element in the ideal. The vanishing loci $V(a)$ and $v(b)$ are both Zariski closed.$V \subseteq V(a) \cup V(b)$ as $ab \in I_k(V)$ and $V = V(I_k(V))$. Since V is irreducible, either $V(a) \supseteq V$ or $V(b) \supseteq V$. WLOG, $V(a) \supseteq V$, i.e., for any $x \in V$, $a(x) = 0$, and $a \in I_k$.

For the other direction, assume that V is $k-$reducible. Write $V = V_1 \cup V_2$, where V_1 and V_2 are proper Zariski closed subsets of V defined over k. Since $V_i \subset V$ is proper, there is $p_i \in I_k(V_i) \setminus I_k(V)$. But $p_1 p_2 \in V$, witnessing that $I_k(V)$ is not a prime ideal. □

Lemma 4.22. *And affine algebraic variety V can be written uniquely as an irredundent union of finitely many irreducible Zariski closed subsets. Likewise for k-irreducibles. Moreover, for k algebraically closed, being $k-$irreducible is equivalent to being absolutely irreducible.*

Proof. Suppose V is an affine algebraic variety defined over k. We are done if V is $k-$irreducible. Suppose V is $k-$reducible, and let V_1, V_2 be $k-$subvarieties of V witnessing this. And continue this process with V replaced by V_1 and V_2. This process must end in finitely many steps by DCC of Zariski closed subsets.

Suppose $V = V_1 \cup \cdots \cup V_r = W_1 \cup \cdots \cup V_s$, where $V_i \subsetneq V$ are distinct irreducible $k-$subvarieties, and similarly for W_i. Suppose $V_i \neq W_k$ for any $1 \leq k \leq s$. Fix $I \subseteq \{1, ..., s\}$ minimal such that $V_i \subseteq \bigcup_{i \in I} W_i$. Then $V_i = (V_i \cap W_{i_0}) \cup (V_i \cap \bigcup_{i \in I, i \neq i_0} W_i)$ shows that V_i is reducible, a contradiction. Thus for any $1 \leq i \leq r$, there is exactly one $1 \leq j \leq s$ such that $V_i = W_j$, and vice-versa.

Assume that V is $k-$irreducible for an algebraically closed k. Assume towards a contradiction that $V = V_1 \cup V_2$ witness V being absolutely reducible. Let $f \in k[\bar{x}], f_1, \ldots, f_r, g_1, \ldots g_s \in \overline{K}(\bar{[}x])$ such that $V = V(f)$, $V_1 = V(f_1, \cdots, f_r)$ and $V_2 = V(g_1, \ldots, g_s)$. Note that it is first order in L_r to say that $V = V_1 \cup V_2$ and $V_1, V_2 \subsetneq V$, using the polynomials f, f_i, g_j. Since $k \prec \overline{K}$ in L_r, there are $k-$definable V_1' and V_2' witnessing V being $k-$irreducible. □

Proposition 4.23. *Fix $k < \overline{K}$. There is a one-to-one correspondence between complete n-types $p(\bar{x}) \in S_n(k)$ and $k-$irreducible affine varieties $V \subseteq \overline{K}^n$.Given $p(\bar{x}) = \mathrm{tp}(\bar{a}/k)$, write $V(p) = V(I(\bar{a}/k))$. Given $k-$irreducible variety V, write*

$$\begin{aligned} p_v(\bar{x}) &= \textit{generic type of } V/\,k \\ &= \{P(\bar{x}) = 0 : P \in I_k(V)\} \cup \{Q(\bar{x}) \neq 0 : Q \notin I_k(V)\}. \end{aligned}$$

Moreover, $\dim(p) = \dim(V(p))$*, where the latter is the dimension in terms of Zariski topology. We call p_v the generic type of V, and $V(p)$ the locus of $\bar{a}$ over k, where $\bar{a} \models p$.*

Proof. Given $p(\bar{x}) \in S_n(k) = \mathrm{tp}(\bar{a}/k)$, want to show that $\mathfrak{a} = I(\bar{a}/k)$ is a prime ideal. Indeed, if $P, Q \in k[\bar{x}]$ such that $PQ \in \mathfrak{a}$, then $PQ(\bar{a}) = 0$, and either $P(\bar{a}) = 0$ or $Q(\bar{a}) = 0$. So one of P or Q must be in $\mathfrak{a}$.

Conversely, fix $k-$irreducible affine variety V. Need to show p_V is consistent. Suppose not, then there are $Q_1(\bar{x}), \ldots, Q_n(\bar{x}) \notin I_k(V)$, with n minimal such that

$$\{P(\bar{x}) = 0 : P \in I_k(V)\} \models Q_1(\bar{x}) = 0 \vee \cdots \vee Q_n(\bar{x}) = 0.$$

So for all Q_i, $V(I_k(V), Q_i)$ are proper non-empty Zariski closed subsets of V, contradicting V being $k-$irreducible. □

$S_n(k)$ corresponds to the collection of $k-$irreducible varieties over k. It has a weaker topology, namely the Zariski topology, where the closed sets are

$$\{p(\bar{x}) \in S_n(k) : p(\bar{x}) = \text{tp}(\bar{a}/k) \text{ for } \bar{a} \in W\},$$

where W is a variety over k.

Discussion of elementary/Naive algebraic geometry: We have the Zariski topology on an affine algebraic variety $V \subseteq \overline{K}^n$, where the closed sets are given by the closed subsets of V. $V^m \subseteq \overline{K}^{nm}$ is also an affine variety, so it has its own Zariski topology, which is distinct from the product topology.

Given a $k-$irreducible variety $V \subseteq \overline{K}^n$ over k, its coordinate ring over k

$$k[V] = k[x_1, \ldots, x_n]/I_k(V)$$

is an integral domain. Its elements give polynomial maps

$$V \to \overline{K}$$

defined over k. The quotient $k(V)$ field of $k[V]$ is called the function field of V over k. Given a rational function $f \in k(V)$, $f :\to \overline{K}$ is defined on a Zariski open set.

Given two $k-$ irreducible affine varieties over K $V \subseteq \overline{K}^n$ and $W \subseteq \overline{K}^m$, a morphism $f : V \to W$ defined over k is a map whose coordinates are in $k[V]$. A $k-$rational map $f : V \to W$ is a map whose coordiates are in $k(V)$. V and W are isomorphic over k if there are morphisms $f : V \to W$ and $g : W \to V$ over k which are mutual inverses. V and W are birationally equivalent over k if there are non-empty Zariski open sets $U \subseteq V$ and $U' \subseteq W$ defined over k and $k-$rational maps $f : V \to W$ and $g : W \to V$ which induce mutual inverses between U and U'.

A main problem of algebraic geometry is the classification of algebraic varieties up to birational equivalence. This problem can be translated into model theory.

Given V, W as before, let $p_V(\bar{x})$ and $p_W(\bar{x})$ be generic types of V and W over k. Then V and W are birationally equivalent over k if and only if their realizations, $\bar{a}$ of p_V and $\bar{b}$ of p_W are interdefinable over k, if and only if $k(V)$ and $k(W)$ are isomorphic over k as fields.

Other Varieties:

A Zariski open subset of an affine variety is a **quasi-affine** variety. E.g., $\bar{x} \in V \wedge P(x) \neq 0$. These have their own induced Zariski topology and coordinate rings $k[V]$.

$$\mathbb{P}^n(k) = \text{ the set of lines through origin in } k^{n+1}.$$

$$\mathbb{P}^n(k) = \{[a_0 : \cdots : a_n] : \exists i\, a_i \neq 0\},$$

where $[a_0 : \cdots : a_n] = [b_0 : \cdots : b_n]$ if there is $0 \neq \lambda$ such that $a_i = \lambda b_i$ for all i. A **projective variety** is a subset of $\mathbb{P}^n(k)$ given by the vanishing of finitely many homogeneous polynomials. $\mathbb{P}^n(\mathbb{C})$ withe the euclidean topology is compact. The algebraica geometric analogue of compactness is completeness. (See [13]) A **quasiprojective variety** is a Zariski open subset of a projective variety.

Most generally, an abstract variety is given by the collection of data

$$X = X_1 \cup \cdots \cup X_n$$

, irreducible affine varieties V_i for $i = 1, \ldots, n$ and bijections $f_i : X_i \to V_i$, such that the transition maps $f_i \circ f_j^{-1}$ are isomorphisms between Zariski opens.

$\mathbb{P}^n(K)$ is an abstract variety with charts given by

$$X_i = \{[a_0, \ldots, a_n : A_i = 1]\},\ i = 0, \ldots, n$$

and bijections

$$X_i \to K^n$$

by omitting a_i.

An algebraic group is an algebraic variety G equipped with a group structure $m : G \times G \to G$, where m is given by a morphism. An algebraic group is connected if and only if it is irreducible. The irreducible components of an algebraic group are the connected components, and are the cosets of the connected component of the identity, which is a subgroup denoted by G^0.

There are two extreme cases.

1. G is linear, i.e.,

   $$G \leq \mathrm{GL}_n(K)$$

   defined by a polynomial equation. $\mathrm{GL}_n(K)$ is, on the face of it, Zariski open in K^n given by $\det(\bar{x}) \neq 0$. But we identify $\mathrm{GL}_n(K)$ with

   $$\{(\bar{x}, \tfrac{1}{\det(\bar{x})}) : \bar{x} \in \mathrm{GL}_n(k)\} \subseteq K^{n+1},$$

   which is Zariski closed. So linear groups are affine. But it is a theorem that an affine algebraic group is linear.

2. G is connected and projective, then G is commutative. Such a G is called an abelian variety, and $\dim(G) = 1$ if and only if G is an elliptic curve.

By the way, we can talk about the various algebraic groups being defined over k.

Theorem (Decomposition Theorem)**.** *For any connected algebraic group G, we have a connected linear group N and an abelian variety A such that*

$$1 \to N \to G \to A \to 1$$

is a short exact sequence of algebraic groups. As a consequence, any algebraic group is a quasiprojective variety.

Theorem. *Any definable connected group G (in $\overline{K}$) is definably isomorphic to an algebraic group.*

This depends on a theorem of Andre Weil's. A connected group G can be recovered from *birational data* defined over K. G has a unique generic type $p(\bar{x})$ over K. For $\bar{a}\bar{b}$ realizing p with $\bar{a} \downarrow_K \bar{b}$, we have $\bar{a} \cdot \bar{b}$ also realizes p.

For a, b, c realizing p independently over K,

$$(a \cdot b) \cdot c = a \cdot (b \cdot c).$$

There is a generically defined operation in a stationary type $p(\bar{x})$. The key point is that if G is a definable over K connected group, then G has a generic type $p(\bar{x})$ over K, and we obatin a K rational map m on $p \otimes p$ which is generically associative, so Weil gives an algebraic group H.

This can be done in a more generic context, including geometric structures $\overline{M} \models T$. Given $X \subseteq \overline{M}^n$ definable over A, we have a partial type $\Sigma(\bar{x})$ over $\overline{M}$ of generics in X over $\overline{M}$, i.e.,

$$\{p(\bar{x}) \in S_x(\overline{M}) : p(\bar{x}) \text{ has maximal dimension in } X\}$$

is a closed set, and is a definable partial type. Suppose we have some $\overline{M}-$definable $f(-,-)$ such that for $a \downarrow b$, $a, b \models \Sigma$, $f(a, b) \models \Sigma$ and is associative, then we get a definable group G with generics $\Sigma(\bar{x})$.

4.2 Basics of $\mathbf{DCF}_0$

DCF_0 is the model companion of the theory DF_0 of differential fields of characteristics 0 in the language

$$L_{r,\partial} = \{+, \times, -, 0, 1, \partial\},$$

and it is ω-stable. The case of several commuting derivations, $\mathrm{DCF}_{0,m}$, is also $\omega-$stable and have interesting open questions. All rings we talk about here are commutative, with 1, and of characteristics 0.

Definition 4.24. A derivation on a ring R is an additive map

$$\partial : R \to R$$

which satisfies the Lebniz law,

$$\partial(xy) = \partial(x)y + x\partial(y).$$

The (ring of) constants C_R of R is

$$\{x \in R : \partial(X) = 0\}.$$

Example. Rings and fields of functions.

- Let $U \subseteq \mathbb{C}$ be an open connected domain. Let R be the ring of holomorphic functions $f(z) : U \to \mathbb{C}$. R is a differential ring with $\partial = \frac{d}{dz}$.
- $\mathbb{C}(z)$ and $\mathbb{C}(z, e^z)$ are differential fields with $\partial = \frac{d}{dz}$.

Note: Given a differential ring (R, ∂) and $A \subseteq R$, the subring of R generated by

$$\{\partial^n(a) : a \in A, n \in \mathbb{N}\}$$

is the smallest differential subring of R containing A.

Differential algebraic geometry (DAG) a lá Kolchin, is the study of solution sets of systems of differential polynomials.

Definition 4.25. Given a differential field (k, ∂), the differential polynomial ring in n indeterminants is

$$K\{x_1, \dots, x_n\} = K[x_1, \dots x_n, \partial(x_1), \dots, \partial(x_n), \dots, \partial^{(r)}(x_j)].$$

Remark 4.26. 1. $k\{x_1, \dots, x_n\}$ becomes a differential ring with the derivation extending ∂ by $\partial' \restriction_k = \partial$ and

$$\partial'(\partial^{(n)}(x_i)) = \partial^{(n+1)}(x_i)$$

for all $i = 1, \dots, n$. We will write ∂ for ∂'.

2. Let $(k, \partial) < (L, \partial)$ be differential fields. Let $P(x_1, \dots, x_n) \in k\{x_1, \dots, x_n\}$. Then we can evaluated P at any $(a_1, \dots, a_n) \in L^n$.

Lemma 4.27. *1. Let (R, ∂) be a differential ring which is an integral domain. Then ∂ extends uniquely to a derivation on the fraction field of R.*

2. Let (k, ∂) be a differential field. Then ∂ has a unique extension to a derivation on k^{alg}.

Proof. 1. Exercise.

2. Let $k \leq k^{alg} = L$. Let $a \in L \setminus k$, and $P(x)$ the minimal (monic) polynomial of a over k. Say

$$P(x) = x^n + b_{n-1}x^{n-1} + \cdots + b_a x_x + x_0.$$

Compute

$$\begin{aligned} \partial(P(x)) &= nx^{n-1}\partial(x) + \partial(b_{n-1})x^{n-1} + b_{n-1}(n-1)x^{n-2}\partial(x) + \cdots + \partial(b_1)x_1 + b_1\partial(x) + \partial(b_0) \\ &= \frac{d}{dx}(P(x))\partial(x) + P^{\partial}(x) \end{aligned}$$

where $P^{\partial}(x)$ denotes the result of applying ∂ to the coefficients of P. Plug in a for x, we get

$$0 = \partial(P(a)) = \frac{d}{d}(P(a))\partial(a) + P^{\partial}(a).$$

As P is the minimal polynomial, $P'(a) \neq 0$, so we can define

$$\partial(a) = -\frac{P^{\partial}(a)}{P'(a)}.$$

□

Remark 4.28. 1. A main thing here is computing $\partial(P(x))$. This generalizes to computing $\partial(P(x_1, \dots, x_n))$ for $P(x_1, \dots, x_n) \in k[x_1, \dots, x_n]$. I.e.,

$$\partial(P(x_1, \dots, x_n)) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} P(x_1, \dots, x_n) \partial x_i + P^{\partial}(x_1, \dots, x_n).$$

2. This is close to the definition of the tangent bundle/space $T(V)$ of an algebraic variety $V \subseteq k^n$, where $k \models \mathrm{ACF}_0$. I.e., suppose V is an irreducible variety over k, $I_k(V)$ is generated by polynomials $P_1, \dots, P_r$, then $T(V) \subseteq k^{2n}$ in $(x_1, \dots, x_n, u_1, \dots, u_n)$ is given by

$$P_1(x_1, \dots, x_n) = 0, \dots P_r(x_1, \dots, x_n) = 0$$

and

$$\sum_{j=1}^{n} \frac{\partial}{\partial x_j} P(x_1, \dots, x_n) u_j = 0, \text{ for } i = 1, \dots, n. \qquad (\star)$$

Given $\bar{a} \in V$, $T(V)_{\bar{a}}$ is given by the linear equations in $(\star)$ at $\bar{x} = \bar{a}$, i.e.

$$\sum_{j=1}^{n} \frac{\partial}{\partial x_j} P(a_1, \dots, a_n) u_j = 0$$

Let DF_0 be the theory of differential fields of characteristic 0.

Definition 4.29. Let $K \leqslant L$ be differential fields and let $a \in L$. We say that a is differentially algebraic over K if $Q(a) = 0$ for some non-zero $Q(x) \in K\{x\}$. Otherwise a is differentially transcendental over K.

Note that a quantifier-free formula in $L_{r,\partial}$ is equivalent modulo DF_0 to a Boolean combination of formulas of form $Q(\bar{x}) = 0$, where $Q(\bar{x}) \in \mathbb{Q}\{x_1, \dots, x_n\}$, i.e. formulas of form $P(\bar{x}, \partial(\bar{x}), \dots, \partial^s(\bar{x})) = 0$ where $P \in \mathbb{Q}[\bar{x}_0, \dots, \bar{x}_s]$. Given a tuple $\bar{a}$ in a differential field, we let $\nabla_r(\bar{a})$ denote the tuple $(\bar{a}, \partial(\bar{a}), \dots, \partial^r(\bar{a}))$, and we let $\nabla_\infty(\bar{a})$ denote the (infinite) tuple $(\bar{a}, \partial(\bar{a}), \partial^2(\bar{a}), \dots)$. So we get the following:

Remark 4.30. Let $K \leqslant L$ be differential fields and let $\bar{a}$ be a finite tuple from L. Then $\mathrm{qftp}_{L_{r,\partial}}(\bar{a}/K)$ is determined by $\mathrm{qftp}_{L_r}(\nabla_\infty(\bar{a})/K)$.

Proposition 4.31. *Let K be a countable differential field. Then for each $n \in \omega$ there are only countably many complete quantifier-free n-types over K. (Recall that, by a complete quantifier-free n-type over K, we mean a collection of formulas of the form $\mathrm{qftp}_{L_{r,\partial}}(\bar{a}/K)$ for some n-tuple $\bar{a}$ in a differential field $L \geqslant K$.*

Proof. It is enough to deal with the case $n = 1$, since $\mathrm{qftp}(\bar{a}_1, \bar{a}_2/K)$ is determined by $\mathrm{qftp}(\bar{a}_1/K)$ and $\mathrm{qftp}(\bar{a}_2/K\langle\bar{a}_1\rangle)$, where $K\langle\bar{a}_1\rangle$ is the differential field generated by K and $\bar{a}_1$. So consider a differential field $L \geqslant K$ and an element $a \in L$.

Case 1: Suppose a is differentially transcendental over K. Then $\mathrm{qftp}_{L_r}(\nabla_\infty(a)/K)$ is uniquely determined; it is the type of an countably infinite algebraically independent tuple K. So by Remark 4.30 also $\mathrm{qftp}_{L_{r,\partial}}(a/K)$ is uniquely determined.

Case 2: Suppose a is differentially algebraic over K. So there exists n such that $\nabla_n(a)$ is an algebraically dependent tuple over K. Pick n minimal possible, and write $a_i = \partial^i(a)$ and $\bar{a} = (a_0, \ldots, a_n)$. Let $P(x_0, \ldots, x_n)$ be an irreducible polynomial over K such that $P(a_0, \ldots, a_n) = 0$. As in the proof of 4.27, we can check that $\partial(a_n) = -P^\partial(\bar{a})/\frac{\partial P}{\partial x_n}(\bar{a}) = 0$. So $\partial^{n+1}(a)$ lies in $K(a_0, \ldots, a_n)$, whence $\partial^k(a)$ lies in $K(a_0, \ldots, a_n)$ for all k, and so the quantifier-free type of a/K is determined by P.

So there is a unique quantifier-free type of a differentially transcendental element, and the quantifier-free type of a differentially algebraic element is determined by a natural number n and a polynomial in $K[x_0, \ldots, x_n]$. Thus there are only countably many quantifier-free types over K. □

Remark 4.32. The above proposition says that (any completion of) DF_0 is 'quantifier-free ω-stable'. This is enough to obtain ordinal-valued Morley rank for quantifier-free formulas; see for example recent work of Pillay, Point, and Rideau-Kikuchi.

Now let us turn to axioms for DCF_0. Classically there are the 'Blum axioms'. Later Pierce and Pillay gave the so-called 'geometric axioms', which are in terms of D-varieties, and later Fornasiero and Terzo showed that it is enough to include only the geometric axioms for affine space. We will discuss the geometric axioms later, but for now we will just give the Blum axioms.

Given $f \in K\{x\}$, the 'order' of f is the largest m such that $\partial^m(x)$ appears non-trivially.

Definition 4.33. ('Blum axioms') DCF_0 is the $L_{r,\partial}$-theory containing DF_0 and containing the following axiom for every $m < n$: for any differential polynomials $f(x), g(x)$ in one variable with $\mathrm{ord}(g) = m, \mathrm{ord}(f) = n$, there is some a with $f(a) = 0$ and $g(a) \neq 0$.

Proposition 4.34. DCF_0 *is a complete consistent theory. It is the model companion of* DF_0*, and it has quantifier elimination.*

Proof. We first check that every model of DF_0 embeds in some model of DCF_0. So let $K \models DF_0$. By a standard 'union of chains' argument, it suffices to show that, for every $f(x), g(x) \in K\{x\}$ with $\mathrm{ord}(g) < \mathrm{ord}(f)$, there is a differential field $L \geqslant K$ with $\mathrm{ord}(g) < \mathrm{ord}(f)$. By 4.27, we may assume without loss of generality that K is algebraically closed. Let f have order n, and write $f(x) = P(x, \partial(x), \ldots, \partial^n(x))$, where $P(\bar{x}) \in K[x_0, \ldots, x_n]$; replacing $P(\bar{x})$ with some irreducible factor of it in which x_n appears non-trivially, we may assume that P is irreducible over K. So

$R := K[\bar{x}]/\langle P\rangle$ is an integral domain. Let $a_0, \ldots, a_n$ denote the images of $x_0, \ldots, x_n$ in R, so that $R = K[a_0, \ldots, a_n]$. (We say $\bar{a} = (a_0, \ldots, a_n)$ is a 'generic' solution of $P(\bar{x}) = 0$.) Let $L_0 \geqslant K$ be the field of fractions of R. We will extend the derivation ∂ on K to a derivation (that we also call ∂) on L_0. First notice that, since x_n appears non-trivially in P and P is irreducible, $a_0, \ldots, a_{n-1}$ are algebraically independent over K. Define $\partial a_i = a_{i+1}$ for $i \leqslant n-1$, and define ∂a_n to be $-P^{\partial}(\bar{a})/\frac{\partial P}{x_n}(\bar{a})$. (So $a_n \in K(\bar{a}) = L_0$, not in $K[\bar{a}]$.) More generally, given $h(a_0, \ldots, a_n) \in K[\bar{a}]$, define $\partial(h(a_0, \ldots, a_n))$ to be $h^{\partial}(\bar{a}) + \sum_{i=0}^{n} \frac{\partial h}{\partial x_i}(\bar{a})a_{i+1}$. Now extend ∂ to L_0 by the quotient rule. We leave it to the reader to verify that these rules give a well-defined derivation ∂ on L_0. Now $f(a_0) = P(a_0, \ldots, a_n) = 0$, so that a_0 is a solution of $f(x) = 0$. On the other hand, since g has order $< n$, $g(a_0)$ is of form $Q(a_0, \ldots, a_{n-1})$ for some $Q \in K[x_0, \ldots, x_{n-1}]$ and since $(a_0, \ldots, a_{n-1})$ is algebraically independent over K also $g(a_0) \neq 0$, as needed.

So, indeed DCF_0 is a consistent theory, and every model of DF_0 embeds in a model of DCF_0.

Now we verify QE, for which we do a back-and-forth argument between ω-saturated models $K, L \models DCF_0$. Suppose we have $f : K_0 \to L_0$ an isomorphism between finitely generated differential subfields of K and L respectively. Suppose $a \in K$. If a is differentially transcendental over K_0, then by the axioms and ω-saturation we find $b \in L$ also differentially transcendental over L_0, and f extends to an isomorphism $K_0\langle a\rangle \to L_0\langle b\rangle$ since $f(\text{qftp}(a/K_0)) = \text{qftp}(b/L_0)$. Otherwise a is differentially algebraic over K_0. Then let

$$n = \text{ord}(a/K_0) := \min\{n : \nabla_n(a) \text{ algebraically independent over } K_0\}.$$

Let $P \in K_0[x_0, \ldots, x_n]$ be irreducible such that $P(\nabla_n(a)) = 0$. Let $Q = f(P)$. The axioms and ω-saturation given $b \in L$ such that $(b, \partial(b), \ldots, \partial^{n-1}(b))$ are algebraically independent over L_0 and such that $Q(\nabla_n(b)) = 0$. By the proof of Proposition 4.31, $f(\text{qftp}(a/K_0)) = \text{qftp}(b/L_0)$, and so f again extends to an isomorphism $K_0\langle a\rangle \to L_0\langle b\rangle$.

So DCF_0 indeed has QE and is the model companion of DF_0. To check completeness, just note that any model of DF_0 contains $\mathbb{Q}$ with the trivial derivation as a substructure, so between any two models of DCF_0 there is a partial elementary embedding. □

From 4.31 we have:

Corollary 4.35. DCF_0 *is* ω*-stable.*

So every from Chapter 2 and Chapter 3 of the course applies, including Morley rank, independence, unique prime models over all sets, etc.

Ellis Kolchin used the expression 'constrainedly closed' for differentially closed, and used the term 'differential closure' for the prime model over a differential field K of characteristic 0, which we denote K^{diff}. Following the notation of Kolchin, we will let $\mathcal{U}$ or $(\mathcal{U}, \partial)$ denote our saturated model of DCF_0.

Lemma 4.36. *Let $A \subset \mathcal{U}$ small. Then* $\mathrm{dcl}(A)$ *is* $\mathbb{Q}\langle A\rangle$*, the differential field generated by A, and the model-theoretic algebraic closure* $\mathrm{acl}(A)$ *coincides with the field-theoretic algebraic closure of* $\mathbb{Q}\langle A\rangle$*.*

Proof. Clearly $\mathbb{Q}\langle A\rangle \subseteq \mathrm{dcl}(A)$, so we may assume without loss that $A = k$ is a differential field. Let $a \in \mathcal{U}$. If a is either differentially transcendental over k or differentially algebraic over k of order > 0, then by the axioms and saturation $\mathrm{tp}(a/k)$ has infinitely many realizations. So if $a \in \mathrm{acl}(k)$ then $\mathrm{ord}(a/k) = 0$, ie a lies in the field theoretic algebraic closure of k.

Now suppose that $a \in \mathrm{dcl}(k)$. So in particular by the above $a \in k^{\mathrm{alg}}$. Let $P(x)$ be the (monic) minimal polynomial of a over k. The computation in the proof of 4.27 shows that $P(x) = 0$ isolates (in the model-theoretic sense) $\mathrm{tp}(a/k)$. But $\mathrm{tp}(a/k)$ has a unique realization, so P must have degree 1, i.e. $a \in k$. □

Lemma 4.37. *Let $\mathcal{C}$ be the field of constants of $\mathcal{U}$. Then $\mathcal{C}$ is an algebraically closed field. Moreover, any definable (in $\mathcal{U}$ with parameters) subset of $\mathcal{C}^n$ is quantifier-free-definable with parameters in* $(\mathcal{C}, +, \cdot)$*.*

Proof. Let $a \in \mathrm{acl}(\mathcal{C})$, and let P be the (monic) minimal polynomial of a over $\mathcal{C}$. Then $\partial(a) = -P^{\partial}(a)/P'(a)$. But $P^{\partial} = 0$ since all coefficients are constants. The rest follows from stable embeddedness and QE. Indeed, by stable embeddedness, any definable subset of $\mathcal{C}^n$ is already definable with parameters from $\mathcal{C}$. But, by QE, a subset $X \subseteq \mathcal{C}^n$ definable with parameters in $\mathcal{C}$ is a Boolean combination of formulas of the form $P(\nabla_n(x), \nabla_m(\bar{b})) = 0$ for $\bar{b}$ a tuple from $\mathcal{C}$. But $\nabla_m(\bar{b}) = (\bar{b}, \bar{0}, \ldots, \bar{0})$, and if the definable set is a subset of $\mathcal{C}^n$ then also $\nabla_n(x) = (x, 0, \ldots, 0)$. So we can replace such a formula by $P(x, 0, \ldots, 0, \bar{b}, \bar{0}, \ldots, \bar{0}) = 0$ and get the same set. □

Lemma 4.38. *Let $A \subseteq B \subset \mathcal{U}$ be small parameter sets and let $\bar{a}$ be a finite tuple. Then $\bar{a}$ is independent from B over A iff $\nabla_\infty(\bar{a})$ is independent from $\mathbb{Q}\langle B\rangle$ over $\mathbb{Q}\langle A\rangle$ in* ACF_0*.*

Proof. to fill in □

Proposition 4.39. DCF$_0$ *has elimination of imaginaries.*

Proof. to fill in □

In general, in a stable 'eliminate canonical bases' as in the proof of Proposition 4.39 then we have weak elimination of imaginaries.

Let us make a few remarks on types and ideals. Given a differential field K, consider the differential polynomial ring $K\{\bar{x}\}$. By a differential ideal in $K\{\bar{x}\}$, we mean an ideal in $K\{\bar{x}\}$ closed under ∂. By the Ritt-Raudenbusch basis theorem, we have the ascending chain condition on differential radical ideals in $K\{\bar{x}\}$, i.e. a differential radical ideal is finitely generated as a differential ideal. (In particular, we have the ascending chain condition on prime differential ideals.) In analogy with the Zariski topology, we get the 'Kolchin topology' or 'differential Zariski topology' on $\mathcal{U}^n$, where a Kolchin closed set is something defined by finitely many $P(\bar{x}) = 0$, where $P \in \mathcal{U}\{\bar{x}\}$. We have the differential analogue of the nullstellensatz, namely a correspondence between the Kolchin closed sets and the radical differential ideals, and we have the DCC on Kolchin closed sets. By quantifier elimination, given a differential field $K < \mathcal{C}$, the prime differential ideals in $K\{\bar{x}\}$ correspond to complete types $p(\bar{x}) \in S_n(K)$. And every Kolchin closed set over K is a finite union of K-irreducible components.

Altogether, one has a complete analogy with ACF_0. There is however one important difference, which is that, in ACF_0, (absolute) irreducibility is definable: given finitely many polynomials $P_i(\bar{x}, \bar{y}) \in \mathbb{Z}[\bar{x}, \bar{y}]$, the set of $\bar{b}$ such that $V(\{P_i(\bar{x}, \bar{b}) : i\})$ is irreducible is definable in ACF_0. The analogue fails in DCF_0.

Now let us discuss rank and dimension in DCF_0. Let $K < \mathcal{U}$ be a differential field and consider $p(\bar{x}) = \mathrm{tp}(\bar{a}/K) \in S_n(K)$. We define $\mathrm{ord}(p) = \mathrm{tr.deg}(K(\nabla_\infty(\bar{a}))/K)$, which is ∞ or finite. We have also the ordinal $\mathrm{RM}(p)$. Finally, we also have the ordinal-valued U-rank $U(p)$, which is defined inductively:

1. $U(\mathrm{tp}(\bar{a}/K)) = 0$ if $\mathrm{tp}(\bar{a}/K)$ is algebraic.
2. $U(\mathrm{tp}(\bar{a}/K)) \geqslant \gamma$ for a limit ordinal γ if $U(\mathrm{tp}(\bar{a}/K)) \geqslant \alpha$ for all $\alpha < \gamma$.
3. $U(\mathrm{tp}(\bar{a}/K)) \geqslant \alpha + 1$ if there exists a differential field $L > K$, such that $\mathrm{tp}(\bar{a}/L)$ forks over K and $U(\mathrm{tp}(\bar{a}/L)) \geqslant \alpha$.

We will show that one of these ranks is finite if and only if they all are.

Let's focus first on 1-types. In particular let's compute $\mathrm{RM}(x = x)$.

Lemma 4.40. *Fix a complete type $p(x) \in S_1(K)$. If $\mathrm{ord}(p)$ is finite, then $U(p) \leqslant \mathrm{RM}(p) \leqslant \mathrm{ord}(p)$.*

Proof. fill in □

Lemma 4.41. *There exist formulas $\phi(x)$ in one variable of arbitrarily large finite Morley rank.*

Proof. Note that each formula $\partial^n(x) = 0$ defines a vector space $V_n \subseteq \mathcal{U}$ over $\mathcal{C}$, with $V_0 = \{0\}$ and $V_1 = \mathcal{C}$. Moreover, ∂ gives a $\mathcal{C}$-linear map $V_{n+1} \to V_n$ with kernel $\mathcal{C}$. So each V_n has $\mathcal{C}$-dimension n. In particular, after adding parameters naming a basis, V_n is in definable bijection with $\mathcal{C}^n$. But $\mathcal{C}^n$ has Morley rank n by using Lemma 4.37 and computing the Morley rank in ACF_0. Hence also V_n has Morley rank n. □

Corollary 4.42. $\mathrm{RM}(x = x) = \omega$ *and there exists a unique complete* 1*-type (over any set) of Morley rank ω: the type of a differentially transcendental element.*

Proof. By 4.41 $\mathrm{RM}(x = x) \geqslant \omega$. By QE, any definable subset (with parameters) of $\mathcal{U}$ is a Boolean combination of equations of form $P(x, \partial(x), \ldots, \partial^n(x)) = 0$, with P a polynomial over some parameters. But any such formulas has finite order and hence finite Morley rank by 4.40.

Hence, for any definable set $X \subseteq \mathcal{U}$, either X or $\mathcal{U} \setminus X$ has finite Morley rank. So we cannot partition $\mathcal{U}$ into two definable sets of infinite Morley rank, whence $\mathrm{RM}(x = x) = \omega$ and $\mathrm{dM}(x = x) = 1$. □

(We can also show that the unique type of Morley rank ω has U-rank ω. Hence if $U(p)$ is finite, then p is not the generic type, whence $\mathrm{ord}(p) < \omega$.)

Lemma 4.43. *Let K be a differential field, s.t. C_K are alg. closed. Then $C_{K^{\mathit{diff}}} = C_K$.*

Proof. Suppose $a \in K^{\mathrm{diff}}$, and $\partial(a) = 0$. Suppose we find $a \in \mathrm{acl}(K)$. Then we know $P(a) = 0$ for some nonzero polynomial over K. We may assume $P =$ minimal monic polynomial of a over K. If P has degree 1, then $a \in K$. Otherwise, $\partial(P(a)) = P'(a)\partial(a) + P^{\partial}(a) = 0$. So $P^{\partial}(a) = 0$. Since P^{∂} has smaller degree, $P^{\partial}(x) = 0$, so $P(x) \in C_K(x)$. So $a \in \mathrm{acl}(C_K)$, so $a \in C_K$.

Otherwise, $\mathrm{tp}(a/K)$ is isolated by $\varphi(x)$ which has infinitely many solutions and $\varphi(x) \vdash \partial(x) = 0$. Since $\mathscr{C}$ is strongly minimal, $\varphi(x)$ defines a cofinite subset of $\mathscr{C}$. So $\varphi(x)$ is realized in C_K. Contradiction. □

Remark. This also shows $C_{K^{\mathrm{diff}}} = C_K^{\mathrm{alg}}$.

Chapter 5

Functional transcendence and $\mathbf{DCF}_0$

What is functional transcendence?

Consider a base field K such as $\mathbb{C}$ or $\mathbb{C}(t)$, and some field F of functions containing K, e.g. $F =$ meromorphic function over some domain $U \subseteq \mathbb{C}$.

Let $\bar{f}$ be a finite tuple for F. What is $\operatorname{tr.deg}(K(\bar{f})/K)$?

Note: "usual" transcendence theory is a bout $\operatorname{tr.deg}(\mathbb{Q}(\bar{a})/\mathbb{Q})$, where $\bar{a}$ finite tuple from $\mathbb{C}$. (Note: $\operatorname{tr.deg}(\mathbb{Q}(e, \pi)/\mathbb{Q})$ is open).

So $\mathbb{Q}$ is replaced by something like $\mathbb{C}(t)$.

In algebraic geometry, there was a big theme to actually, proving things in $\mathbb{C}(t)$, analogues of $\mathbb{Q}$ might not be so hard, but going to $\mathbb{F}_p(t)$ is interesing. Charactersitic p is interesting.

Function field Mordell-Lang, in char $= 0$ case, Anand found another proof, in these ams notes.

Schanuel conjecture: Suppose $x_1, \ldots, x_n \in \mathbb{C}$ are $\mathbb{Q}$-linearly independent, then

$$\operatorname{tr.deg}(\mathbb{Q}(x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n})/\mathbb{Q}) \geq n.$$

Special case is where $x_1, \ldots, x_n \in \mathbb{Q}^{\text{alg}}$, so the Schanuel conj. says $\operatorname{tr.deg}(\mathbb{Q}(e^{x_1}, \ldots, e^{x_n})/\mathbb{Q}) = n$. This is Lindemann's theorem.

So the idea is that any algebraic relations, come from multiplicative group.

Ax-Schanuel: function field version of Schanuel. Proved by Ax.

Also an Ax-Lindemann version.

So minimum we want to prove Ax-Lindemann by using the binding group.

Definition 5.1. Let (K, ∂) be differential field, V an K-irr. affine variety over K. ($V \subseteq n$-space). The shifting tangent bundle $\tau(V)$ (or $T_\partial(V)$) is the affine variety defined by $\{P(x) = 0 :$

$P \in I_K(V)\} \cup \{\sum_{i=1}^{n} \left(\frac{\partial}{\partial x_i} P(\bar{x})\right) \bar{v} + P^{\partial}(\bar{x}) = 0 : P \in I_K(V)\}$. In fact, it's enough to use finitely many $P_1, \ldots, P_m$ generating $I_K(V)$.

If $I_K(V)$ generated by polynomials over C_K, then $\tau(V) = T(V)$.

Suppose $\bar{a} \in V$, $\bar{a} \in (L, \partial) > (K, \partial)$, then $(\bar{a}, \partial(\bar{a})) \in \tau(V)(L)$. You can think of it as some Zariski closure of $(\bar{a}, \partial(\bar{a})) : \bar{a} \in V$. (But not always).

We have $\tau(V) \to V$ projection to first n coordinates.

Theorem 5.2. *Let $K \models DCF_0$. $V \subseteq K^n$ an irr. variety. Because K is alg. closed, irr. is absolutely irr. Let $W \subseteq \tau(V)$ be irreducible variety, s.t. W projects dominantly onto V. Let $U \subseteq W$ be nonempty Zariski open. Then there is $(\bar{a}, \bar{b}) \in U(K)$, s.t. $\partial(\bar{a}) = \bar{b}$.*

Remark 5.3. The statement of theorem 5.2 + K is algebraically closed, also axiomatize DCF_0.

We make use of an extension theorem for deviation. See X.7.7 of [7].

Fact 5.4. Let K be a differential field. $L = K(a_1, \ldots, a_n)$. Let $V = V(\bar{a}/K)$. the variety over K defined by zeros of all polynomials over K vanishing at $\bar{a}$.

Let $W = \tau(V)$. Suppose $\bar{b} = (b_1, \ldots, b_n) \in L$ s.t. $(\bar{a}, \bar{b}) \in W$. Then ∂ extends to a derivation $\partial : L \to L$, s.t. $\partial(\bar{a}) = \bar{b}$.

Proof. Sketch proof of 5.2: (See [9] for more details): Let $(\bar{a}, \bar{b})$ be a generic over K point in W in some field L containing K. We will apply Fact 5.4 with W in place of W. Consider $\tau(W)_{(\bar{a}, \bar{b})}$. Then $\tau(W)_{\bar{a}, \bar{b})} = \{(\bar{b}, \bar{w}) : \bar{w}$ satisfies a given constant finite set of linear equations over $\bar{a}, \bar{b}\}$. For $\bar{c} \in K(\bar{a}, \bar{b})$, s.t. $(\bar{a}, \bar{b}, \bar{b}, \bar{c}) \in \tau(W)$, By lemma 5.4, $\exists$ an extension of the derivation ∂ over K to ∂^* on $K(\bar{a}, \bar{b})$, s.t. $\partial^*(\bar{a}, \bar{b}) = (\bar{b}, \bar{c})$. So in particular $\partial^*(\bar{a}) = \bar{b}$.

As DCF_0 has QE and model companion of DF_0, we can embed (L, ∂^*) over (K, ∂) into $(U, \partial) \models \text{DCF}_0$. (We may assume $(L, \partial^*) \subseteq (U, \partial)$).

Since $(\bar{a}, \bar{b})$ is generic point on W and $(\bar{a}, \bar{b}) \in U$ and $\partial(\bar{a}) = \bar{b}$, and $K \prec U$, we can find such $(\bar{a}_1, \bar{b}_1) \in U(K)$, with $\partial(\bar{a}_1) = \bar{b}_1$.

□

Definition 5.5. Let K be a differential field. By a homogeneous n-dim linear differential equation over K (in vector form) we mean $\partial Y = AY \quad (*)$ where $Y = (y_1, \ldots, y_n)^t$ indeterminate, A an $n \times n$ matrix over K and ∂ is applied coordinatewise.

Remark 5.6. 1. We can talk about the solution of $(*)$ in K, a set V of column vectors, so subset of K^n which is

(a) A subgroup of $(K^n, +)$.

(b) A C_K-subspace of $(K^n, +)$.

2. A homogenous scalar linear differntial equation (DE) of order n is something of the form $y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$ where $a_i \in K$, $y^{(n)} = \partial^{(n)}(y)$.

   This is a special case of linear DE in vector form. Why?

   Consider

$$\partial(y_0, \ldots, y_{n-1})^t = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & \cdots & \cdots & -a_{n-1} \end{pmatrix} (y_0, \ldots, y_{n-1})^t$$

   Then the solutions of the equations are precisely of form

$$(y, \partial(y), \ldots, \partial^{(n)}(y))^t$$

   where y is the solution of the scalar order n equation.

Lemma 5.7. *Let K be a differential field. $\partial Y = AY$ and K as before* (dim $= n$)*. Let $Y_1, \ldots, Y_m$ be solution (in K^n). Then $Y_1, \ldots, Y_m$ are linearly independent over K iff they are linearly independent over C_K.*

Proof. Conversely suppose $Y_1, \ldots, Y_m$ are linearl dependent over K. We may assume m is leas, s.t. $Y_1, \ldots, Y_m$ are linearly dependent over K. So we have $a_2, \ldots, a_m \in K$, s.t.

$$Y_1 + a_2 Y_2 + \cdots + a_m Y_m = 0 \qquad (*).$$

Apply ∂ to both sides, we get

$$AY_1 + a_2' Y_2 + a_2 A Y_2 + \cdots + a_m' Y_m + a_m A Y_m = 0. \qquad (**)$$

Multiply $(*)$ by A and substract from $(**)$ we get

$$a_2' Y_2 + \cdots + a_m' Y_m = 0.$$

So $a_i' = 0$, $i = 2, \ldots, m$. So $a_i \in C_K$. □

So by lemma 5.7, if $\partial Y = AY$, is an n-dim linear DE over K, and $V =$ solution space of K, then $\dim_{C_K} V \leq n$.

Proposition 5.8. *Let $K \models DCF_0$. $\partial Y = AY$ n-dim linear DE over K. V solution space. Then* $\dim_{C_K}(V) = n$.

Proof. Consider $X = K^{n^2}$ an affine irr. variety over K. Then $\tau(X) = T_\partial(X) = T(X) = K^{2n^2}$. Identify K^{n^2} with $n \times n$ matrices over K.

Then $W = \{(\bar{x}, A\bar{x}) : \bar{x} \in K^{n^2}\}$ is an irreducible subvariety of K^{2n^2} projecting onto X. Then $\{(\bar{x}, A\bar{x}) : \bar{x} \in K^{n^2},\ \det(\bar{x}) \neq 0\}$ is a Zariski open of W projecting dominantly onto X.

So by theorem 5.2 applied to X, $\exists Z \in \mathrm{GL}_n(K)$, s.t. $\partial Z = AZ$. So columns of Z form a C_K-basis for solution space of $\partial Y = AY$. □

Remark 5.9 (Important remarks)**.** K differential closed.

1. Given $\partial Y = AY$ n-dim linear DE ove K. Let $(Y_1, \ldots, Y_n)$ be a C_K-basis for solution space V of $\partial Y = AY$. By 5.7, $(Y_1, \ldots, Y_n)$ is a solution of the equation $\partial Z = AZ$ on GL_n, i.e. a solution of $\partial Z \cdot Z^{-1} = A$. This is a log derivative on GL_n.

2. The differential alg. map $\partial Z \cdot Z^{-1}$ on $\mathrm{GL}_n(K)$ is a "crossed" homeomorphism $\mathrm{GL}_n(K) \to \mathfrak{gl}_n(K)$. This map is also called $\mathrm{dlog}_{\mathrm{GL}_n}$.

   (Γ is a group that G acts on. $\alpha : G \to \mathrm{Aut}(\Gamma)$ is the action. A crossed homomorphism from G to Γ is a function $\phi : G \to \Gamma$ that satisfies $\phi(g_1 g_2) = \phi(g_1) \cdot \alpha(g_1)(\phi(g_2)), \forall g_1, g_2 \in G$).

3. Let $G < \mathrm{GL}_n(K)$— be an algebraic subgroup (connected) defined over C_K. Then

   $$\mathrm{dlog}_{\mathrm{GL}_n} \restriction G : G(K) \to LG(K) = \mathrm{dlog}_G.$$

   Where LG is Lee algebra.

   For example, denote $\mathbb{G}_m$ be the multiplicative group (of whatever field we are thinking about). Then $\mathbb{G}_m \times \ldots \times \mathbb{G}_m$ We can view it as subgroup of GL_n living on the diagonal:

   $$G(K) = \left\{ \begin{pmatrix} a_1 & \cdots & 0 \\ 0 & \cdots & a_n \end{pmatrix} \in \mathrm{GL}_n(K) : a_i \in K. \right\}$$

   Then

   $$\mathrm{dlog}_{\mathrm{GL}_n} \restriction G \left(\begin{pmatrix} a_1 & \cdots & 0 \\ 0 & \cdots & a_n \end{pmatrix} \right) = \begin{pmatrix} a_1'/a_1 & \cdots & 0 \\ 0 & \cdots & a_n'/a_n \end{pmatrix}$$

   So $\mathrm{dlog}_G(a_1, \ldots, a_n) = (a_1'/a_1, \ldots, a_n'/a_n)$.

4. For every connected algebraic group G over C_K, we have $\mathrm{dlog}_G : G(K) \to LG(K)$ (Geometric understanding of dlog_G for $G = \mathbb{G}_m$.)

Now $K = \mathbb{C}(t)$. $(\mathbb{C}, +) \xrightarrow{\exp} (\mathbb{C}^*, \times)$. Denote $\mathbb{G}_a(\mathbb{C}) = (\mathbb{C}, +)$ and $\mathbb{G}_m(\mathbb{C}) = (\mathbb{C}^*, \times)$. What is $y \mapsto y'/y$ on $\mathbb{G}_m(K)$? For $x(t) \in \mathbb{G}_a(K)$, $\exp(x(t))$ is a function $y(t) \in \mathbb{G}_m(L)$ on $(\mathbb{C}^\times, \times)$, where L is space of meromorphic functions on $\mathbb{G}_m$.

$y(t) = \exp(x(t))$. Then $y'(t) = d/d(t)(y'(t)) = x'(t)\exp(x(t)) = x'(t)y(t)$. So $y'/y = x'$.

Now we connect the linear DE's with internality and bitorsors, etc. from chapter 3.

Now let K be a differential field, not necessarily $\models DCF_0$, s.t. C_K is alg. closed. (e.g. $\mathbb{C}(t), d/dt$). Let $K \subseteq U$ living in a model of DCF_0. And $\partial Y = AY$ a linear DE over K, of $\dim = n$.

Let V = subspace in U, vector space over $\mathscr{C}$. So V is internal to $\mathscr{C}$. The machineary from chapter 3 comes into play.

In particular $\mathrm{Aut}(V/K, \mathscr{C})$ is definable. Also we get a bitorsor (see remark 3.17).

We make it more explicit: by 5.7 we reduce to finding a "fundamenntal matrix of solutions", i.e. a solution of $\partial Z = AZ$ on GL_n, i.e. of $\partial Z \cdot Z^{-1} = A$. This is called a log differential equation on GL_n over K.

A lot more about $\mathrm{dlog}_{\mathrm{GL}_n} = \partial Z \cdot Z^{-1}$: it is a croosed homomorphism from $\mathrm{GL}_n(U)$ to $\mathfrak{gl}_n(U)$, i.e. $\mathrm{dlog}(gh) = \mathrm{dlog}(g) + (\mathrm{dlog}(h))^g$. Then

$$\begin{aligned}\ker(\mathrm{dlog}) &= \{g : \mathrm{dlog}(g) = 0\} \\ &= \mathrm{GL}_n(\mathscr{C}).\end{aligned}$$

The solution Y of $\partial Z Z^{-1} = A$ is a left coset $b \cdot \mathrm{GL}_n(\mathscr{C})$ of the kernal. So if another solution $b_1 \in Y$, then $b_1 \in b \cdot \mathrm{GL}_n(\mathscr{C})$, so $b^{-1}b_1 \in \mathrm{GL}_n(\mathscr{C})$.

Let b be a solution of $\partial Z = AZ$ in K^{diff}, i.e. $b \in \mathrm{GL}_n(K^{\mathrm{diff}})$. By the way, $K\langle b\rangle = K(b)$ is called the Picard-Vessiot (PV) extension of K for $\partial Y = AY$.

NB: if b_1 is another solution of $\partial Z = AZ$ in K^{diff}, then $b^{-1}b \in \mathrm{GL}_n(C_{K^{\mathrm{diff}}}) = \mathrm{GL}_n(C_K)$. So $K\langle b_1\rangle = K\langle b\rangle$.

Let $q = \mathrm{tp}(b/K)$. Let Q be the set of realizations in U. Then q is isolated, fundamental $\mathscr{C}$-internal, and weakly orthonogal to $\mathscr{C}$, i.e. q implies a complete type over $K \cup \mathscr{C}$. [If not, $\exists$ realization b_1, b_2 of $q(y)$ and formula $\varphi(y, \bar{w})$ over K, and $\bar{c} \subseteq \mathscr{C}$, s.t. $\varphi(b_1, \bar{c}) \wedge \neg\varphi(b_2, \bar{c})$. Let $\psi(y)$ isolate q. So $U \models \exists y_1 \exists y_2 \exists \bar{w}(\psi(y_1) \wedge \psi(y_2) \wedge \varphi(y_1, \bar{w}) \wedge \neg\varphi(y_2, \bar{w}) \wedge \bar{w} \in \mathscr{C}$. So we can find $y_1, y_2, \bar{w}$ in K^{diff}. So $\bar{w} \in C_K$. $y_1, y_2 \models q$, contradicting q isolated.]

So remark 3.17 holds, giving us a bitorsor (G, Q, H) defined over K with $H \subseteq \mathscr{C}^{\mathrm{eq}}$.

Let's be more explicit, also working in K^{diff} rather than U. (Point by homogeneity of K^{diff}, we could still useargument by automorphism).

Redefine Q = realization of q in K^{diff}. (Fix a copy of K^{diff}. So Q is a solution of $\mathrm{GL}_n(K^{\mathrm{diff}})$ and a left coset of $\mathrm{GL}_n(C_K)$.) We consider $\mathcal{G} = \mathrm{Aut}(Q/K) = \mathrm{Aut}(Q/K, C_{K^{\mathrm{diff}}})$ = elementary permutations of Q induced by $\mathrm{Aut}(K^{\mathrm{diff}}/K)$.

Fix $b \in K^{\mathrm{diff}}$ and $b \in Q$. Note for any $\sigma \in \mathrm{Aut}(Q/K)$, $\exists$ unique $c_\sigma \in \mathrm{GL}_n(C_K)$ s.t. $\sigma(b) = b \cdot c_\sigma$, i.e. $b^{-1} \cdot \sigma(b) = c_\sigma$.

Define $H = \{b^{-1} \cdot b_1 : b_1 \in Q\} \leq \mathrm{GL}_n(C_K)$. only dependent on $\mathrm{tp}(b/K)$. Because it is $= \{b_2^{-1}b_1 : b_1, b_2 \in Q\}$. ($Y$ = solution set of $\partial Z = AZ$, $Y \subseteq \mathrm{GL}_n(U)$, Y is left coset of $\mathrm{GL}_n(\mathscr{C})$. $Q \subseteq Y(K^{\mathrm{diff}})$.

Q left coset of H. H is a definable (in $(C_K, +, \cdot)$) subgroup of $\mathrm{GL}_n(C_K)$. So an algebraic group Q, H. This is the left hand side.

right hand side: define $G = \{\sigma(b) \cdot b^{-1} : \sigma \in \mathrm{Aut}(Q/K)\} \leq \mathrm{GL}_n(K^{\mathrm{diff}}$. Which doesn't dependent on $b \in Y$. Let $b_1 \in Y$. $b_1 = b \cdot d$, $d \in \mathrm{GL}_n(C_K)$. Then

$$\sigma(b_1)b_1^{-1} = \sigma(b \cdot d)(bd)^{-1} = \sigma(b)dd^{-1}b^{-1} = \sigma(b)b^{-1}.$$

Then $\sigma \mapsto \sigma(b)b^{-1}$ gives an isomorphism of $\mathrm{Aut}(Q/K)$ action on Q to G with left multiplication on Q. Bitorsor: (G, Q, H). G is definable over K subgroup of $\mathrm{GL}_n(K^{\mathrm{diff}})$ definably isomorphic to H over K, b.

Remark 5.10. 1. We can identify $\mathrm{Aut}(L/K)$, where $L = K(b)$, with $\mathrm{Aut}(Q/K)$. $L = K(b_1)$ for any $b_1 \in Q$.

So $\mathrm{Aut}(L/K)$ has structure of linear algebraic group in the contents.

2. There is a Galois correspondence between intermediate differential fields $K \leq L' \leq L$ and algebraic subgroups of H.

Lemma 5.11. *With notation above,* $\mathrm{tr.deg}(K(b)/K) = \dim H = RM(H)$ *in either* K^{diff} *or in* $(C_K, +, \cdot)$.

Proof. Exercise. □

Notice in K^{diff}, $\mathrm{RM}(Q) = \mathrm{RM}(H)$. Q isolates a type over K in $\mathrm{tp}(g/K)$. So $\mathrm{RM}(Q) = \mathrm{RM}(\mathrm{tp}(b/K))$. $\mathrm{tr.deg}(K(b)/K) =$ order of $\mathrm{tp}(b/K)$. Why it's $= RM(\mathrm{tp}(b/K))$? Left to readers.

Now, let us recall Lindemann's theorem: $x_1, ..., x_n \in \mathbb{Q}^{alg}$ $\mathbb{Q}$-linearly independent implies that $\exp(x_1), ..., \exp(x_n) \in \mathbb{C}$ are algebraically independent.

Recall that $\exp z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$. We note that exp is a group homomorphism from $(\mathbb{C}, +)$ onto $(\mathbb{C}^*, \times)$ (in fact, the former is the universal cover of the latter), and so $\exp(q_1 x_1 + ... + q_n x_n) = \exp x_1{}^{q_1} ... \exp x_n{}^{q_n}$ for $q_i \in \mathbb{Q}$.

In the function field version, $\mathbb{Q}^{alg}$ is replaced by K^{alg} (where $K = \mathbb{C}(t)$, and the derivation is $\frac{d}{dt}$). If $x \in K^{alg}$, it lies in some finite extension of K, so we can view x as a rational function on a complex algebraic curve S. Then $\exp(x)$ corresponds to a meromorphic function on some disc in S. Let us write $\exp x \in F > \mathbb{C}(S)$, where F is the field of meromorphic functions on some disc on S. In this context, we can state the Ax-Lindemann theorem:

Theorem 5.12. *Let* $x_1, ..., x_n \in K^{alg}$ *be* $\mathbb{Q}$*-linearly independent modulo* $\mathbb{C}$ *(i.e. there are no* $q_1, ..., q_n \in \mathbb{Q}$ *not all 0 such that* $q_1 x_1 + ... q_n x_n \in \mathbb{C}$*).*

1. *Suppose* F *is a differential field extending* K^{alg} *and* $y_1, ..., y_n \in F$ *such that* $\frac{\partial y_i}{y_i} = \partial x_i$ *for* $i = 1, .., n$. *Then* $y_1, ..., y_n$ *are algebraically independent over* K.

2. *In particular, if* $y_i = \exp(x_i)$, *then* $y_1, ..., y_n$ *are algebraically independent over* K^{alg}.[1]

Proof. It is enough to prove 1. The relevant algebraic group G is the algebraic torus $(\mathbb{G}_m)^n$. Then $dlog_G(y_1, .., y_n) = (\partial y_1/y_1, ... \partial y_n/y_n)$. In fact, if we consider $(\mathbb{G}_m)^n$ as the subgroup of diagonal matrices in GL_n, then $dlog_G = \partial Z \cdot Z^{-1}|_G$. For simplicity, we assume that $x_1, .., x_n \in K = \mathbb{C}(t)$. Let $a_i = \partial x_i$, so that $a_i \in K$. Then it is easy to see that any given $\bar{y} = (y_1, ..., y_n)$ as in the statement of the theorem is a solution of the system $dlog_G(\bar{y}) = (a_1, ..., a_n)$. So WNTS $y_1, ..., y_n$

[1] We can obtain this as follows: if $y_i = \exp x_i$, then apply the derivation $\frac{d}{dt}$ to both sides + chain rule to get $\partial y_i = y_i \partial x_i$; then apply part 1.

are algebraically independent over K. Without loss of generality, we may assume that $\bar{y} \in K^{diff}$. Then it follows that $q = \text{tp}(\bar{y}/K)$ is isolated. Let $Q =$ set of realizations of q. By the preceding discussion of internality in the context of differential fields, there exists a bitorsor (G, Q, H) defined over K such that $G \cong \text{Aut}(Q/K) = \text{Aut}(L/K)$ where $L = K(\bar{y})$, and $H \leq GL_n(\mathbb{C})$. Recall the explicit construction of H: for each $\sigma \in \text{Aut}(Q/K)$, $\sigma(y_i)$ is another solution of $\partial y/y = a_i$, so $\sigma(y_i) = y_i b_i(\sigma)$ where $b_i(\sigma) \in \mathbb{G}_m(\mathbb{C})$. It follows from this that:

$$H = \{(b_1(\sigma), ..., b_n(\sigma)) : \sigma \in \text{Aut}(Q/K)\} \leq G \leq GL_n(\mathbb{C})$$

as algebraic subgroups. By Lemma 5.11, tr.dg.$(K(\bar{y}/K) = \dim H$. So if $y_1, ..., y_n$ are not algebraically independent over K, then tr.dg.$(K(\bar{y})/K) < n$, so $\dim H < n = \dim G$, so H is a proper algebraic subgroup of $\mathbb{G}_m(\mathbb{C}) = G$. We now use a fact from the theory of algebraic tori: the algebraic subgroups of $\mathbb{G}_m^n$ are defined by finite systems of equations $z_1^{k_1}...z_n^{k_n} = 1$ for $k_1 \in \mathbb{Q}$ not all 0 (there are no infinite families of definable subgroups — it is *rigid*). In particular, there are $k_1, ..., k_n \in \mathbb{Q}$ not all 0 such that $z_1^{k_1}...z_n^{k_n} = 1$ for all $z_1, ..., z_n \in H$. By the explicit description of H, for all $\sigma \in \text{Aut}(L/K)$, $b_1(\sigma)^{k_1}...b_n(\sigma)^{k_n} = 1$. Now, let $y = y_1^{k_1}...y_n^{k_n}$. Then it follows that $\sigma(y) = y$ for all $\sigma \in \text{Aut}(L/K)$. Since $y \in L$ and y is fixed by all $\sigma \in \text{Aut}(L/K)$, we have that $y \in K$ by homogeneity. But we have that $\frac{\partial y_i^{k_i}}{y_i^{k_i}} = k_i a_i$, so $\partial y/y = k_1 a_1 + ... + k_n a_n = \partial(k_1 x_1 + ... + k_n x_n)$. Letting $x = k_1 x_1 + ... + k_n x_n$, it follows that $\partial y/y = \partial x$. So we have found $x, y \in \mathbb{C}(t)$ such that $\partial y/y = \partial x$. Then by comparing the residues of poles, or by observing that if $f'/f = g'$ and $\log f = g + C$ then $f(t) = Be^{g(t)}$, we deduce that g must be constant. It follows that x must be constant, so $\partial x = 0$, hence $k_1 x_1 + ... + k_n x_n \in \mathbb{C}$. ☐

We remark that dlog_G has a more geometric interpretation as well. Let dlog_G be the logarithmic derivative for G an algebraic group, and K some function field such as $K = \mathbb{C}(t)$ or more generally $\mathbb{C}(S)$ for S an algebraic curve. If X is a variety over $\mathbb{C}$, then $X(K) =$ the set of rational functions $f : S \to X$ over $\mathbb{C}$. Let G be a complex algebraic group and $x \in G(K)$. Then x can be regarded as a rational function $x : S \to G$, i.e. an S-point, so we can think of the differential ∂x as picking out, for each $s \in S$, a tangent vector to G above $x(s)$. It follows that $\partial(x)(s) \in T_{x(s)}G$, and the differential of multiplication by $x(s)^{-1}$ takes $\partial(x)(s)$ to a tangent vector in LG. Then the composition of these two maps is $dlog_G$, which is a map from S to LG.

We now turn to a corollary of Ax-Lindemann:

Theorem 5.13. *Let $K = \mathbb{C}(T)$. Let $x_1, .., x_n \in K^{alg}$, for F a differential field containing K and $y_1, ..., y_n \in F$ s.t. $\partial y_i/y_i = \partial x_i$. Then either $y_1, ..., y_n$ are algebraically independent over K or $\exists k_1, ..., k_n \in \mathbb{Q}$ such that $y_1^{k_1}...y_n^{k_n} \in \mathbb{C}$.*

Proof. Assume $y_1, ...y_n$ are algebraically dependent. In the proof of the previous theorem, we used Galois theory (where the Galois group was H, a proper subgroup of $\mathbb{G}_m^n$), to find $k_1, ..., k_n$ not all 0 such that $y = y_1^{k_1}...y_n^{k_n} \in K$. We also saw that $\partial y/y = \partial x$ for $x = k_1 x_1 + ... + k_n x_n$. Then, as in the proof the previous theorem, we have that $y \in \mathbb{C}$. Conversely, suppose $y_1^{k_1}...y_n^{k_n} \in \mathbb{C}$ for some $k_1,..,k_n$ not all 0. So $\partial y/y = 0$, hence $\partial x = 0$, and so it follows that $k_1x_1 + ... + k_nx_n \in \mathbb{C}$ and $x_1,..,x_n$ are $\mathbb{Q}$-linearly dependent modulo $\mathbb{C}$. □

In particular:

Corollary 5.14. *Again $K = \mathbb{C}(t)$, $x_1,...,x_n \in K^{alg}$ so $x_1,..,x_n$ are rational functions on an algebraic curve. Let $y_i = \exp(x_i)$. Then either $y_1,....,y_n$ are algebraically independent over K or $\exists k_1,...,k_n$ no all 0 such that $y_1^{k_1}...y_n^{k_n} \in \mathbb{C}$.*

Now we state the geometric formulation of Ax-Lindemann:

Proposition 5.15. *Let S be an algebraic curve, and $x_1,...,x_n : S \to \mathbb{C}$ rational functions on S. Let $y_i = \exp(x_i)$, which are maps $S \to \mathbb{C}^*$. Then either $y_1,...,y_n$ are algebraically independent over $K = \mathbb{C}(S)$, or the image of S under $\bar{y}$ (for $\bar{y} : S \to (\mathbb{C}^*)^n$) is contained in a coset of a proper algebraic subgroup of $\mathbb{C}^*$.*

Proof. We assume $y_1,...,y_n$ are not algebraically independent over K and we have to prove that $\exists k_1,...,k_n \in \mathbb{Q}$ not all 0 such that $y_1^{k_n}...y_n^{k_n} \in \mathbb{C}^*$ iff the conclusion of the above statement holds. Suppose first that $y_1^{k_n}...y_n^{k_n} = d \in \mathbb{C}^*$. Then the image of S under y is contained in $\{z \in (\mathbb{C}^*)^n : z_1^{k_1}...z_n^{k_n} = d\}$ which is a coset of a proper algebraic subgroup of $(\mathbb{C}^*)^n$. The converse is the same as the proper algebraic subgroups of $(\mathbb{C}^*)^n$ are defined by systems of equations of the form $z_1^{k_1}...z_n^{k_n} = 1$, where not all $k_i = 0$. Thus, it follows that all the cosets are given by systems of equations of the form $z_1^{k_1}...z_n^{k_n} = d$, as desired. □

There are a number of ways to generalize Ax-Lindemann:

1. In S12, S13, S14, we can replace $K = \mathbb{C}(t)$ by the function field of any algebraic variety. Then we can work with several commuting derivatives of d/dt, working in $\text{DCF}_{0,m}$ with m commuting partial derivatives. The Galois theory carries through in this more general context. Likewise in S15, we can replace the curve S with an algebraic variety.

2. We can also replace $\mathbb{G}_m$ by any semiabelian variety defined over $\mathbb{C}$, where a semiabelian variety is a connected commutative algebraic group G fitting into the exact sequence:

$$0 \to \mathbb{G}_m^n \to G \to A \to 0$$

of algebraic groups where A is an abelian variety (an abelian variety is a commutative algebraic group whose underlying variety is projective). In this setting, exp is replaced by $\exp_G : (\mathbb{C}, +)^d \to G(\mathbb{C})$ where dim $G = d$, $(\mathbb{C}, +)^d$ is the Lie algebra of $G(\mathbb{C})$, and the exponential map is the Lie-theoretic exponential map. Then the relevant differential equation is one that replaces $\partial Z.Z^{-1}$ on $\mathbb{G}_m^n$ is with dlog_G. To see this, we first note that $\exp_G$ has a multivalued inverse $\log_G : G(\mathbb{C}) \to \mathbb{C}^d$. Consider a rational function

$f : \mathbb{A}^1 \to G(\mathbb{C})$. Then $d/dt \log_G(f(t))$ becomes single-valued; this is the $dlog_G$ map. The relevant differential equations on G over K are of the form $\text{dlog}_G(y) = a \in G(K)$. This also supports a Galois theory called Kolchin's strongly normal theory, which generalizes the Picard-Vessiot Galois theory for the general linear group. In [12] and [14], Pillay presents a generalization of Kolchin's strongly normal theory where the relevant algebraic groups are not over the constants. In any case, the key feature that enables the generalization to the case of semiabelian varieties is their rigidity: there is no infinite definable family of connected algebraic subgroups.

We now state the Ax-Lindemann theorem for semiabelian varieties:

Proposition 5.16. *(Ax-Lindemann for semiabelian varieties over $\mathbb{C}$; see [2]) Let G be semiabelian over $\mathbb{C}$, and $\dim G = d$. Let S be an irreducible complex algebraic variety, $x : S \to LG = \mathbb{C}^d$ a rational function, and $y = \exp_G(x)$, so that we obtain a meromorphic map $S \to G(\mathbb{C})$. Then either tr.dg.$(\mathbb{C}(S)(y))/\mathbb{C}(S)) = d$ or the image of S under y is contained in a translate/coset of a proper algebraic subgroup $H(\mathbb{C})$ of $G(\mathbb{C})$.*

Connection with Manin-Mumford: The torsion points of $\mathbb{G}_m$ are Zariski dense (i.e. there is no proper irreducible algebraic subvariety that contains the torsion points), since there are infinitely many of them (e.g. the roots of unity). The same hold more generally for any algberaic torus $\mathbb{G}_m^n$, and indeed, for semiabelian varieties G over $\mathbb{C}$. In this context, we have the following characterization of special subvarieties of a semiabelian variety:

Theorem 5.17 (Manin-Mumford)**.** *Let G be a semiabelian variety over $\mathbb{C}$. Let V be an irreducible subvariety of G such that $Tor(G) \cap V$ are Zariski dense in V. Then V is a translate of a connected algebraic subgroup of G. (These are called semiabelian subvarieties).*

There are many proofs of this theorem. Model-theoretically, there are arguments due to Hrushovski, using (the finite rank part of) ACFA and Zilber dichotomy, and Pillay (with Kowalski), using jet-space arguments. Both of these used stability-theoretic arguments. Then Pila-Wilkie introduced a point-counting theorem in the o-minimal setting which was then used by Pila-Zannier to prove Manin-Mumford. Their strategy crucially involved proving and using Ax-Lindemann.

Now, recall that classical Ax-Lindemann is about $\exp_G : \mathbb{C}^d \to G$ for G semiabelian. There are many generalizations and extensions to other transcendental functions, such as the j-function, using o-minimal techniques.

Ax-Schanuel, i.e. Ax's Theorem ([1])

Earlier, we stated (in Prop. S17) Ax-Schanuel in the following form: let $K = (\mathbb{C}(t)^{alg}, d/dt)$ and $F \supset K$ a differential field extension. If $x_1, ..., x_n \in F$ are $\mathbb{Q}$-linearly independent mod $\mathbb{C}$, and y_i satisfy $\partial y_i / y_i = \partial x_i$ for all i, then tr.dg.$(K(x_1, ..., x_n, y_1, ..., y_n)/K) \geq n$.

This applies to the case where e.g. F is a suitable field of meromorphic functions on a disc and $y_i = \exp(x_i)$.

This follows from the slightly more general statement:

Proposition 5.18. *Let F be any differential field containing $\mathbb{C}$, which is the field of constants of F. Let $x_1, .., x_n, y_1, ..., y_n \in F$ s.t. $x_1, ..., x_n$ are $\mathbb{Q}$-linearly independent mod $\mathbb{C}$ and $\partial y_i / y_i = \partial x_i$. Then tr.dg.$(\mathbb{C}(x_1, ..., x_n, y_1, ..., y_n)/\mathbb{C}) \geq n + 1$.*

Ax originally proved this theorem. Kirby proved this for semiabelian varieties G in [6]. The proof combines ideas from differential algebra and algebraic geometry. There is an o-minimal proof of Ax-Schanuel in the case of algebraic tori by Tsimerman in [16]. For other transcendental functions, there are proofs of appropriate statements in [3], where a general Ax-Schanuel theorem is proven using the theory of connections on principal bundles. One challenge is to give a differential-algebraic (i.e. model theory of DCF$_0$) proof of Ax's theorem and the newer results in [3].

The problem is that, in DCF$_0$, the set defined by $\partial y_i / y_i = \partial x_i$ $(i = 1, ..., n)$ is infinite-dimensional (in terms of Morley rank). But next time, we will discuss a case where this is tractable——namely, the case where $x_1, ..., x_n$ are differentially-algebraically transcendental, i.e. $x_1, x_1', x_1'', x_1''', ..., x_2, x_2', x_2'', ...$ are all algebraically independent.

Now we turn to some remarks on Kirby's approach to Ax-Schanuel (which comes from section 3 of the above paper). Let $S = \mathbb{G}_m^n$ and $LS = \mathbb{G}_a^n$. Then the tangent bundle is $TS = LS \times S$, and the main lemma is the following:

Lemma 5.19. *Suppose F is a differential field, C its constants, which are algebraically closed, and $(\bar{x}, \bar{y}) \in TS(F)$ for $\bar{x} = (x_1, .., x_n), \bar{y} = (y_1, ..., y_n)$ such that $\partial y_i / y_i = \partial y_i$ for all $i = 1, ..., n$ (we will indicate this system of equations by $\partial \bar{y} / \bar{y} = \partial \bar{x}$). Suppose tr.dg.$(C(\bar{x}, \bar{y})/C) \leq n$. Then there is a proper algebraic subgroup $H \leq S$ and $\gamma \in TS(C)$ such that $(\bar{x}, \bar{y}) \in \gamma \cdot TH$ (where $TH \leq TS$).*

The proof goes through differential forms and Kähler differentials. We denote by $\Omega(F/C)$ the F-vector space of Kähler differentials, which is the module generated freely by symbols of the form da for $a \in F$ subject to the conditions: (i) $d\alpha = 0$ for $\alpha \in C$, (ii) d is additive, and (iii) the Leibniz law $d(ab) = adb + bda$. It is the dual space to the F-vector space Der(F/C) of derivations $F \to F$ that vanish on C. We now outline the main steps of the proof:

Step 1: We have fixed $(\bar{x}, \bar{y})$ (which for convenience we will denote (x, y)). Let $\omega_i(x, y) = dy_i / y_i - dx_i \in \Omega(F/C)$. Then under our assumptions, the $\omega_i(x, y)$ are F-linearly dependent.

Step 2: The $\omega_i(x, y)$ are C-linearly dependent, i.e. there are $\alpha_i \in C$ s.t. $\sum_{i=1}^n \alpha_i \omega_i(x, y) \in \Omega(F/C) = 0$.

Step 3: Let $\eta = \sum_{i=1}^n \alpha_i \omega_i$. This is a nonzero invariant differential form on TS over C.

The evaluation of (invariant) differential forms on TS at a point gives a group homomorphism $TS(F) \to \Omega(F/C)$ so evaluating η at (x, y) gives such a homomorphism, and the kernel of μ is a proper subgroup of $TS(F)$.

Step 4: There is a proper algebraic subgroup H of S over C contained in K, and $\omega \in TS(C)$, such that $(x, y) \in \gamma \cdot TH$ (this step uses the Borel/Zilber indecomposability theorem).

Now let us give a differential-algebraic proof of this result in a special case. We work in DCF_0.

Suppose $x_1, ..., x_n \in F$, where F is a differential field, C is the field of constants, and $x_1, ..., x_n$ are $\mathbb{Q}$-linearly independent modulo C. Suppose $\mathrm{tr.dg.}(\bar{x}/C) = 1$ and $C(\bar{x})$ a differential field, with $\partial y_i / y_i = \partial x_i$ for $i = 1, ..., n$. Then $\mathrm{tr.dg.}(C(x, y)/C) \geq n + 1$. There are two extreme cases:

1. $x_1, ..., x_n$ are differentially transcendental and differentially algebraically independent over C.
2. $\mathrm{tr.dg.}(C(\nabla_\infty(\bar{x}))/C) < \omega$

We do the first case. In that case, we prove that if $\partial y_i / y_i = \partial x_i$ for $i = 1, ..., n$, then $\mathrm{tr.dg.}(C(x, y)/C) \geq n + 1$. We remark that any algebraic subgroup of $\mathbb{G}_a^n \times \mathbb{G}_m^n$ is a product (this follows from the fact that there are no nontrivial algebraic group homomorphisms between a vector group and an algebraic torus).

Proof. Assume that the desired conclusion fails. So $\bar{y} \in acl(C, \bar{x})$ so (x, y) is the ACF_0 generic point of an irreducible algebraic variety V over C with $\dim V = n$ (we write (x, y) for tuples $\bar{x}, \bar{y}$). As $x_1, ..., x_n$ are differentially algebraically independent, the U-rank of $\mathrm{tp}(x/C)$ is the Morley rank of $\mathrm{tp}(x/C)$ and this is equal to $\omega \cdot n$. In DCF_0, $U(V) = RM(V) = \omega \cdot n$ and (x, y) is a DCF_0-generic point on V over C. Moreover V is Kolchin-irreducible (by a theorem of Kolchin). Let G be a subgroup of TS defined by $\partial y/y = \partial x$. This is a differential algebraic subgroup. So $U(G) = \omega \cdot n + k$ for some $k < \omega$. As originally $(\bar{x}, \bar{y}) \in G$ and is a DCF_0 generic point of V, we have that $V \subseteq G$. After translating V by a constant point $(\bar{\alpha}, \bar{\beta})$, we have $(\bar{\alpha}, \bar{\beta}) \cdot V \subseteq G$ and this translate contains the identity. By a theorem for superstable groups due to Berline and Lascar, G has a "1-connected" component $H \leq G$ with $U(H) = \omega \cdot n$ (so the quotient has finite Morley rank). As $1 \in V$, it follows that $V = H$ (this is analogous to the case in algebraic groups where for an algebraic group G, any connected component must be a translate of G_0). Alternatively, by the Berline-Lascar version of Zilber-indecomposability, V generates in finitely many steps a definable subgroup of G of U-rank $\omega \cdot n$, and so it must be equal to H. In either case, the conclusion is that $(\bar{\alpha}, \bar{\beta}) \cdot V$ is a definable subgroup of $G \leq TS$, and so $(\bar{\alpha}, \bar{\beta}) \cdot V$ is an algebraic subgroup of TS contained in G. But H projects onto LS (since the projection is a group homomorphism), and H is connected and of dimension n, so $H = LS \times \{id\}$. So it follows that the tuple y (which corresponds to $\bar{\beta}$) is constant, which is a contradiction, since that would imply that the tuple x is constant too. ☐

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